Subspace Topology

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Definition

If X is a topological space with a topology T and Y is a subset of X, then

Ty = {Y ∩ U | U ∈ T} is a subspace topology on Y. The open sets of this topology consist of the intersections of X and Y.

Lemmas and Theorems

Lemma 16.1 If $\mathcal{B}$ is a basis for the topology of X then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is for T_Y

Lemma 16.2 If Y is a subspace of X, U is open in Y, and Y is open in X, then U is open in X.

Theorem 16.3 If A is a subspace of X and B is a subspace of Y, the product topology on A x B is the as the topology passed from X x Y to A x B.

Proof of Lemma 16.1

If $\mathcal{B}$ is a basis for the topology of X then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is for T_Y

Given: U is open in X, $y \in U \cap Y$, Y is a subspace of X, and $\mathcal{B}$ is the basis of the topology on X

1. We choose an element B of $\mathcal{B}$, such that $y \in B \subset U$

2. Since $y \in U \cap Y$ and $B \subset U$, $y \in B \cap Y$ and $B \cap Y \subset U \cap Y$

3. Since, by definition, all open sets of Y consist of the intersections between Y and the open sets of X, $B \cap Y$ and $U \cap Y$ are both open in Y

4. Then for any open set $U \cap Y$ in Y and $y \in U \cap Y$, there exists a $B \cap Y$, such that $y \in B \cap Y \subset U \cap Y$.

5. This implies that $B \cap Y$ is a basis element of Y, and hence the collection of basis elements of the topology on X intersected with Y, does indeed generate the subspace topology on Y and is the basis of the topology on Y.

Proof of Lemma 16.2

If Y is a subspace of X, U is open in Y, and Y is open in X, then U is open in X.

Given: Y is a subspace of X, U is open in Y, and Y is open in X

1. Since U is open in Y, and all open sets of a subspace are generated by the intersection of the subspace itself and the open sets in the topology on its parent, there exists an open set V of the topology on X, such that Y ∩ V = U.

2. Since both Y and V are open in X, and the finite intersection of open sets produces an open set, Y ∩ V is open in X

3. Hence, U is open in X

Proof of Theorem 16.3

If A is a subspace of X and B is a subspace of Y, the product topology on A x B is the same as the topology passed from X x Y to A x B

Given: U is open in X, V is open in Y, A is a subspace of X, B is a subspace of Y, and A x B is a subspace of X x Y.

1. Per the definition of a product topology, some open set U of topology X crossed with some open set V of topology Y is the general basis element of X x Y, and is naturally open in X x Y.

2. Since all open sets of a subspace topology are defined as the intersection of the subspace itself and the open sets of the parent topology, the general basis element of the subspace topology A x B is (U x V) ∩ (A x B)

3. Since A is a subspace of X and B is a subspace of Y, and therefore U ∩ A and V ∩ B are the general basis elements of A and B, respectively, (U x V) ∩ (A x B) = (U ∩ A) x (V ∩ B)

4. (U ∩ A) x (V ∩ B) is the basis of the product topology on A x B

5. Since the bases are fundamentally the same, the two topologies on A x B must be the same.

Exercise

If Y is a subspace of X and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Given: A ⊂ Y, Y is a subspace of X, ~~B is the basis of the topology on X, C is the basis of the topology on Y, and A is also a subspace of Y~~

U is an open set in Y

V is an open set in X

$T_A^Y = \{U \cap A \mid U \in T_Y\}$

$T_A^X = \{V \cap A \mid V \in T_X\}$

Let $r \in U \cap A \Rightarrow r \in U$ and $r \in A$

$U \in T_Y \Rightarrow U = V \cap Y$ for some V of the topology on X

Since $r \in U \cap A$ and $U = V \cap Y$,

$r \in V$ and $r \in Y$

Since $r \in V$ and $r \in A$, $r \in V \cap A$

H $U \cap A$ is an element of