Connected Spaces

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6th Period Topology

Parallels Between

Calculus and Topology

Intermediate Value Theorem

If $f: [a, b] \to \mathbb{R}$ is continuous and if r is a real number between f(a) and f(b), then there exists an element $c \in [a, b]$ such that f(c) = r.

Maximum Value Theorem

If $f: [a, b] \to \mathbb{R}$ is continuous, then there exists an element $c \in [a, b]$ such that $f(x) \leq f(c)$ for every $x \in [a, b]$.

Uniform Continuity Theorem

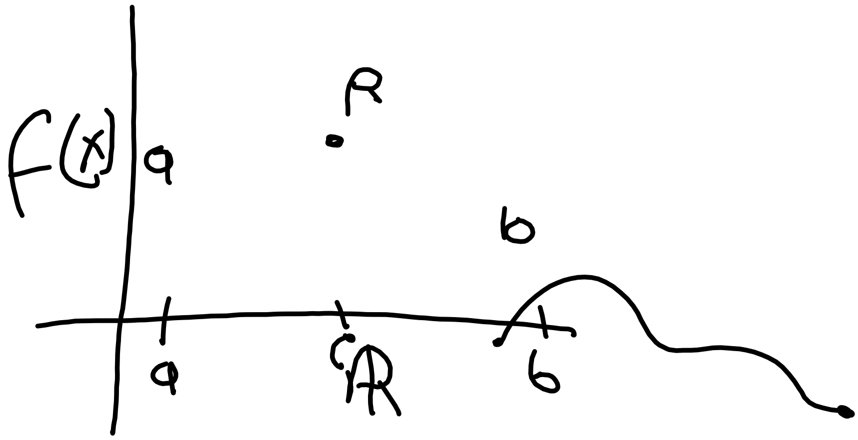
If $f: [a, b] \to \mathbb{R}$ is continuous, then given ϵ > 0, there exists δ > 0 such that $|f(x_1) - f(x_2)| < \epsilon$ for every pair of numbers x_1 , x_2 of [a,b] for which $|x_1 - x_2| < \delta$.

Parallels Between

Calculus and Topology

Intermediate Value Theorem

If $f: [a,b] \rightarrow \mathbb{R}$ is continuous and if r is a real number between f(a) and f(b), then there exists an element $c \in [a,b]$ such that f(c) = r.



Connected Spaces

A space can be "separated" if it can be broken up

into two *disjoint,* open parts. Otherwise it is **connected**.

If the set is not *separated,* it is *connected,* and vice versa.

Connected Spaces

Let X be a topological space. A **separation** of X is a pair of disjoint, nonempty open sets of X whose union is X.

E.g.

$$U \cup V = X$$

$$U, V \neq \emptyset \; and \; are \; open$$

Connected Spaces

A space is connected if and only if

the only subsets of X that are both open and closed

in X are $\emptyset$ *and X itself.*

$$A \subset X$$

$$A \; is \; both \; open \; and \; closed \; in \; X.$$

$$U = A$$

$$V = X - A$$

Subspace Topology

Let X be a topological space with topology $\mathcal{T}$

If Y is a subset of X (I.E. $Y \subset X$),

the collection:

$$\mathcal{T}_y = [Y \cap U | U \in \mathcal{T}]$$

is a topology on X called the Subspace Topology.

Y is called the Subspace, and its open sets consist of all intersections of open sets of X with Y.

Lemma 23.1

If Y is a subspace of X, a separation of Y is a pair of disjoint, nonempty sets A and B whose union is Y, neither of which contain a limit point of the other. The space Y is connected if there exists no separation of Y.

$$Y \subset X$$

$$A \cup B = Y$$

$$A \cap B = \emptyset$$

Let X be a topological space and let $A \subset X$

$x \in X$ is said to be a **Limit Point** of A if every neighborhood of x intersects A in a point other than x.

Let X be a topological space and let $A \subset X$

The **Closure** of A, $\bar{A}$, is the intersection of all closed sets containing A.

Lemma 23.1

If Y is a subspace of X, a separation of Y is a pair of disjoint, nonempty sets A and B whose union is Y, neither of which contain a limit point of the other.

The space Y is connected if there exists no separation of Y.

Let X be a topological space and let $A \subset X$.

The **Closure** of A, $\bar{A}$, is the intersection of all closed sets containing A.

Given: $Y \subset X$

$A \cup B = Y$

$A \cap B = \emptyset$

A, B ⊂ Y

Proof: Let Y be a subspace of X

A & B form a separation of Y.

⇒ A is both open & closed in Y.

⇒ $A = \bar{A} \cap Y$

since A is closed in Y

$\bar{A} \cap B = \emptyset$

∴ A & B are a separation

Lemma 23.4

Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

$A \subset X$

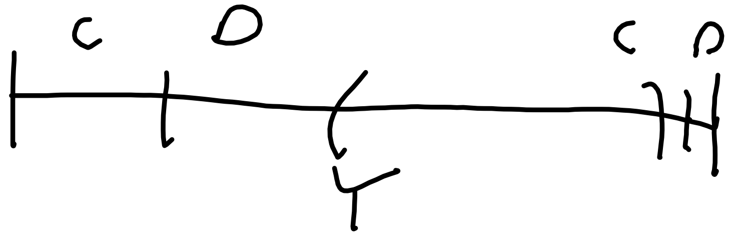
$A \subset B \subset \bar{A}$

Lemma 23.2

If the sets C and D form a separation of X,

and if Y is a connected subspace of X,

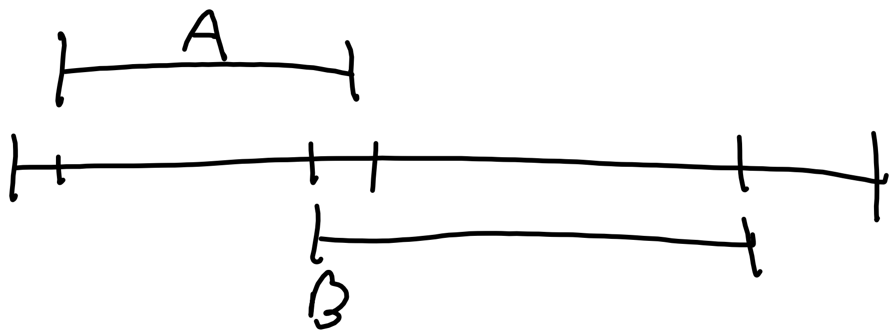
then Y lies entirely within C or D.



Lemma 23.3

The union of a collection of connected subspaces of X

that have a point in common is connected.



Lemma 23.5

A finite cartesian product of connected spaces is connected.

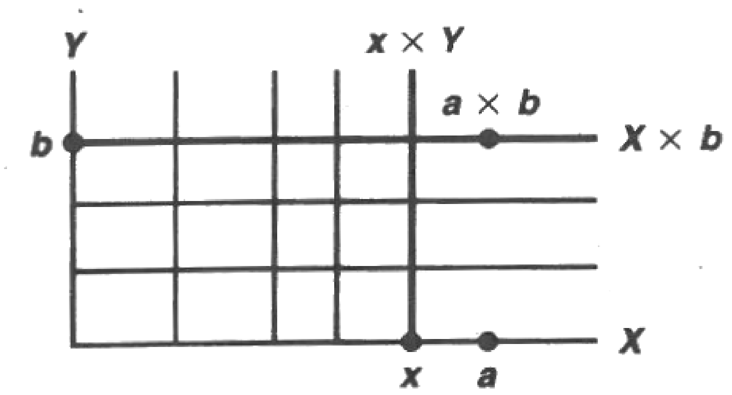


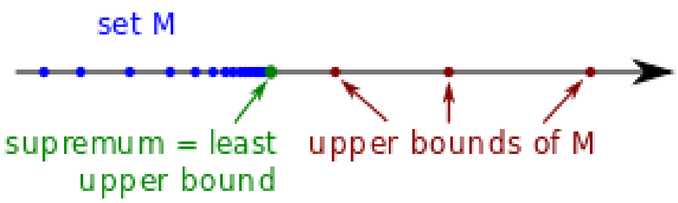
Figure 23.2

Connected Subspaces of the Real Line

A simply ordered set L having more than one element is called a **Linear Continuum** if the following hold:

1) L has the *least upper bound property.*

(See section 3, pg. 27 Munkres)



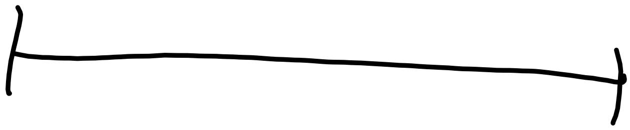
2) If x < y, there exists z such that x < z < y

Connected Subspaces of the Real Line

Theorem 24.1:

If L is a linear continuum in the order topology,

then L is connected and so are intervals and rays in L.

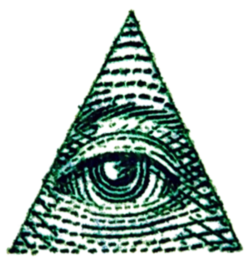


Connected Subspaces of the Real Line

But what does this all mean?

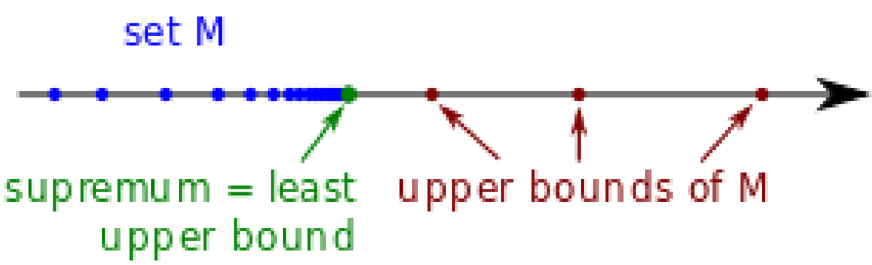
Where can we realistically find a connected space?

Lets find out.



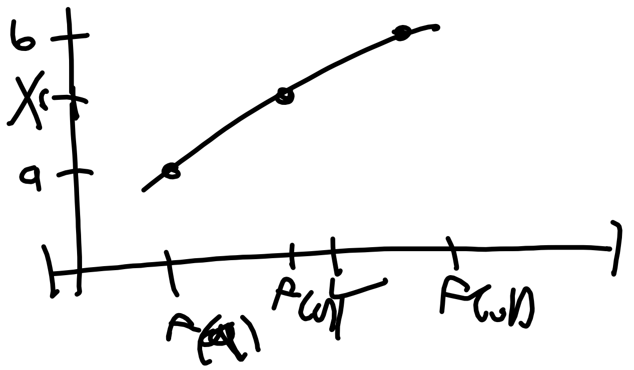
Corollary 24.2

The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.



Connected Subspaces of the Real Line

Let $f: X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between f(a) and f(b), then there exists a point c of X such that f(c) = r.



Path Connectedness

Given points x and y of space X, a **Path** in X from x to y is a continuous map $f: [a, b] \to X$ of some closed interval in the real line ($[a, b] \subset \mathbb{R}$), such that f(a) = x and f(b) = y.

A space X is said to be **Path Connected** if every pair of points of X can be joined by a path in X.