F is said to cover X if X is a topological space and *F* is a family of open subsets of X whose union is all of X. If *F* is said to cover X, and for each *F*- there exist subfamilies *F'* such that $\bigcup F' = X$, the topology X is said to be compact.

The Heine-Borel theorem (Armstrong): A closed interval of the real line is compact

$[a,b]$ $[a,x]$ $x \leq b$

Let $x \in [a,b]$

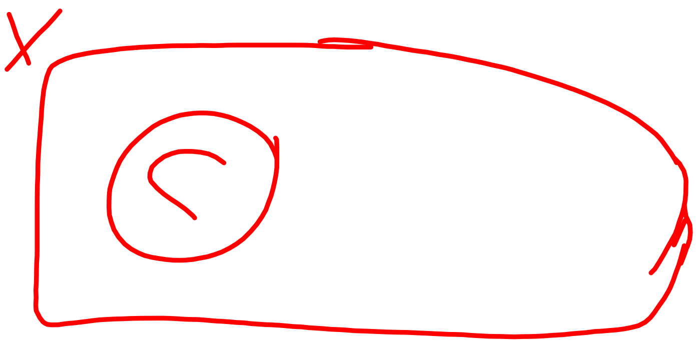
Theorem 3.4 (Armstrong): The continuous image of a compact space is compact

X Z ∈ F such that F covers Y

$f: X \to Y$

Theorem 3.5 (Armstrong): A closed subset of a compact space is compact

Proof: Let X be a compact space, C a closed subset of X, and *F* a family of open subsets of X such that C is an subset of $\bigcup F'$. If we add the open set X – C to *F* we obtain an open cover of X. Using the compactness of X we know that this open cover has a finite subcover. Therefore we can find $O_i \in F$, such that $\bigcup O_i \cup (X - C) = X$. This implies that C is a subset of O_{i-} and the various sets of O provide the required finite subfamily of *F*.



$X = \bigcup F'$

Theorem 3.6 (Armstrong): If A is a compact subset of a Hausdorff space X, and if $x \in X - A$, then there exist disjoint neighborhoods of x and A. Therefore a compact subset of a Hausdorff space is closed.

Proof: Let z be a point of A. Since X is Hausdorff, we can find disjoin sets U_z and V_z such that $x \in U_z$ and $z \in V_z$. We shall vary z in A and the notation is chosen to emphasize the dependence of U_z and V_z on z; remember x is a fixed point of X – A. Varying z throughout a produces a family of open set $\{V_z | z \in A\}$ whose union contains A. But A is compact, so A is a subset of $\bigcup V_z$ for some finite collection of points $\bigcup z \in A$. Let $V = \bigcup V_z$. Since V_{z_i} is disjoint from the open neighborhood U_{z_i} of x, V is disjoint from the intersection $U = \bigcup U_z$. The sets U, V are disjoint open neighborhoods of x and A.

$A = \overline{A}$

Hausdorff space: a topological space is Hausdorff if for each distinct set of points there exist disjoint neighborhoods of x and y

Theorem 3.7 (Armstrong): A continuous bijective function from a compact space X to a Hausdorff space Y is a homeomorphism

Proof: Let f be a function that maps X to Y and let C be a closed subset of X. Then C is compact (theorem 3.5). Therefore f(C) is compact (theorem 3.4) and consequently closed in Y (theorem 3.6). so f takes closed sets to closed sets, which proves that f^{-1} is continuous.

A function is a homeomorphism if both it and its preimage are continuous.

Theorem 27.4 (Munkres): Let a continuous function map X to Y where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

Proof: Since f is continuous and X is compact, Y must be compact (theorem 3.4) containing within its sets a largest element M and a smallest element m, such that m=f(c) and M=f(d). Since both m and M are extreme values of the set, there must exist values m_i between them, such that m_i=f(x).