Theorem: The set of all finite subsets of integers is countable.

hint: prove that $\forall n$, the set of finite subsets of size $n$ is countable.

Recall: a set $S$ is countable if $\exists$ a one-to-one & onto function $f: \mathbb{N} \to S$.

Proof by induction:

set of all subsets of $\mathbb{Z}$ of size 1, call it $A$.

define $g: \mathbb{N} \to A$ by $g(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{-n}{2}+1, & \text{if } n \text{ is even} \end{cases}$

$1 \to 1 \quad 3 \to 2 \quad 5 \to 3 \quad 7 \to 4$

$2 \to 0 \quad 4 \to -1 \quad 6 \to -2 \quad 8 \to -3$ ✓

Suppose the set of all subsets of $\mathbb{Z}$ of size $k$ is countable, call it $B$.

To show that the set of all subsets of $\mathbb{Z}$ of size $k+1$ is countable.

Since $B$ is countable, $\exists\ h: \mathbb{N} \to B$, 1-1 & onto.



Given a set with $n$ objects, the # of subsets containing $k$ elements is $\binom{n}{k}$,

so the total # of subsets of any size is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

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$(1+1)^n = \binom{n}{0}1^n \cdot 1^0 + \binom{n}{1}1^{n-1} \cdot 1^1 + \binom{n}{2} \cdot 1^{n-2} \cdot 1^2 + \dots + \binom{n}{n} 1^0 \cdot 1^n$

$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

$\Rightarrow 2^n$ is the total number of subsets of a set of size $n$

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$B_1 = \{\ldots, \{-3\}, \{-2\}, \{-1\}, \{0\}, \{1\}, \ldots\}$$

$$B_2 \approx B_1 \times B_1$$

$$B_k \approx \underbrace{B_1 \times \cdots \times B_1}_{k \text{ times}}$$

$$B_{k+1} \approx B_k \times B_1$$

If a, then b.

If $\frac{x}{2} + \frac{y}{3} = 1$, then $x^2 + y^2 > 1$.

Contrapositive: If not b, then not a.

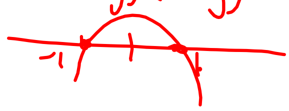
To prove: if $x^2 + y^2 \le 1$, then $\frac{x}{2} + \frac{y}{3} \ne 1$.

If $x^2 + y^2 = 1 \Rightarrow x = \pm\sqrt{1 - y^2}$

$$\frac{x}{2} + \frac{y}{3} = \frac{\pm\sqrt{1-y^2}}{2} + \frac{y}{3}$$

$1 - y^2 \ge 0$

$(1-y)(1+y) = 0$



$-\frac{1}{3} \le \frac{y}{3} \le \frac{1}{3}$ $= \frac{2y \pm 3\sqrt{1-y^2}}{6}$ why is this $\ne 1$?

$-1 \le y \le 1$ for this to be defined

If $y = \pm 1$, this $= \pm 1/3$

If $y = 0$, this $= \pm 1/2$