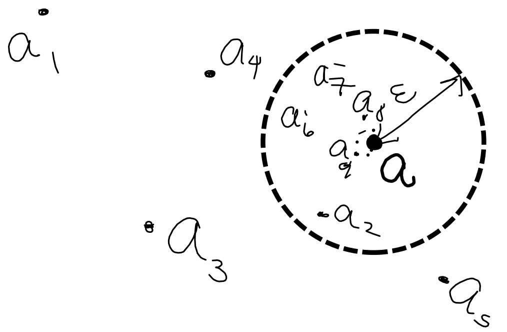
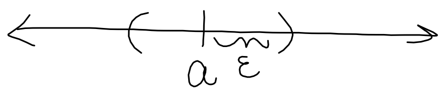
Convergence of a sequence

$a_n \to a$ if given any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon \ \forall n \geq N$.

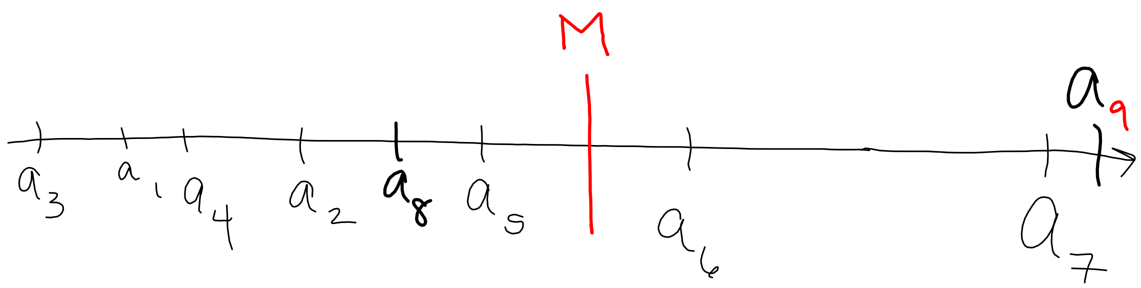




Divergence of a sequence

$\{a_n\}$ diverges if given any $M \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that

$a_n \geq M \ \forall \ n \geq N$.



Cauchy Sequences

$\{a_n\}$ is Cauchy if given any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_m - a_n| < \varepsilon \ \forall m,n \geq N$

2.1
#2 a, c, d
3 a, b, c
4 a, c, d
6
7

2.2
#2, 3, 4, 6, 7

2.4
#1-7

If $a_n \to a$, then $\{a_n\}$ is <u>bounded</u>.

i.e. $\exists\ M \in \mathbb{N}$ s.t. $|a_n| \le M \ \forall n \in \mathbb{N}$.

<u>Thm 2.2.5</u> If $a_n \to a$ & $b_n \to b$

$$\lim_{n\to\infty} (a_n b_n) = ab$$

<u>Proof</u>: Let $\varepsilon > 0$ be given.

$a_n \to a \Rightarrow \exists\ N_1 \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2M}\ \forall n \ge N_1$.

$b_n \to b \Rightarrow \exists\ N_2 \in \mathbb{N}$ s.t. $|b_n - b| < \frac{\varepsilon}{2|a|}\ \forall n \ge N_2$.

We want to find an $N \in \mathbb{N}$ s.t.

$|a_n b_n - ab| < \varepsilon \ \forall n \ge N$.

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab|$$

$\boxed{\triangle \text{ ineq: } |x+y| \le |x| + |y|}$ ☆

$$\le |a_n b_n - ab_n| + |ab_n - ab|$$

$$= |b_n| \cdot |a_n - a| + |a||b_n - b|$$

$b_n \to b \Rightarrow \exists\ M \in \mathbb{N}$ s.t. $|b_n| \le M \ \forall n \in \mathbb{N}$.

$$\le M|a_n - a| + |a||b_n - b|$$

$$= M|a_n - a| \quad \text{if } a = 0$$

We want

$$\underbrace{M|a_n - a| + |a||b_n - b|}_{\le M \cdot \frac{\varepsilon}{2M} + |a| \cdot \frac{\varepsilon}{2|a|}} < \varepsilon$$

$$\le M \cdot \frac{\varepsilon}{2M} + |a| \cdot \frac{\varepsilon}{2|a|}$$

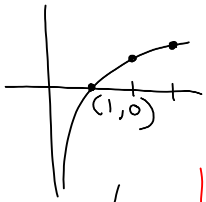
$$= \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\left\{\ln\left(\frac{1}{n}+1\right)\right\} \longrightarrow 0$$

we need to find an $N \in \mathbb{N}$ s.t.

$$\left|\ln\left(\frac{1}{n}+1\right) - 0\right| < \varepsilon \quad \forall\, n \geq N.$$

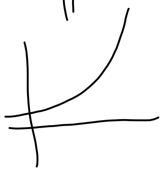
$$\left|\ln\left(\frac{1}{n}+1\right)\right| < \varepsilon$$



since $\frac{1}{n}+1 > 0$

$\ln\left(\frac{1}{n}+1\right) > 0$

$$\ln\left(\frac{1}{n}+1\right) < \varepsilon$$



e^n is strictly increasing

that is if $a<b$, then $e^a<e^b$.

$$\frac{1}{n}+1 < e^{\varepsilon}$$

$$\frac{1}{n} < e^{\varepsilon}-1$$

$$n > \frac{1}{e^{\varepsilon}-1}$$

Take $N = \frac{1}{e^{\varepsilon}-1}$