$p(x)$, polynomial, $a_n \to a$

$\lim_{n\to\infty} p(a_n) = p(a)$

If $p(x) = c$; $\lim_{n\to\infty} p(a_n) = c = p(a)$

If $p(x) = x$

$= cx$

$= c_1 x + c_2 x \cdot x + c_3 x \cdot x \cdot x + \dots$

$p(x) = c_0 x^0 + c_1 x^1 + c_2 x^2 + \dots + c_i x^i$

show $i = 0$, 1

assume $i = k$; show for $i = k+1$

follows from basic limit rules

$$\lim_{n\to\infty}\left(c_0 (a_n)^0 + c_1 (a_n)^1 + c_2 (a_n)^2 + \dots + c_k (a_n)^k\right) = c_0 + c_1 a + c_2 a^2 + \dots + c_k a^k$$

$$\lim_{n\to\infty}\left(\underbrace{c_0 (a_n)^0 + \dots + c_k (a_n)^k}_{=} + c_{k+1} (a_n)^{k+1}\right)$$

$+ c_{k+1} a_n^k \cdot a_n^1$, $a_n^k \to a^k$, $a_n^1 \to a^1$

$+ c_{k+1} a^{k+1}$

$a_n \leq b_n \; \forall n$, $a_n \to \infty$

prove that $b_n \to \infty$

$a_n \to \infty \Rightarrow$ Given any $M \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $a_n \geq M \; \forall n \geq N$.

Since $b_n \geq a_n \; \forall n$, $b_n \geq a_n \geq M \; \forall n \geq N$,

Hence $b_n \to \infty$.

$a_n = 1 + \frac{1}{\sqrt{n}}$ → prove it's Cauchy.

$\{a_n\}$ is Cauchy if given $\varepsilon > 0$ $\exists N > 0$ such that $|a_m - a_n| < \varepsilon$ $\forall m, n \geq N$.

we want $\left|1 + \frac{1}{\sqrt{m}} - \left(1 + \frac{1}{\sqrt{n}}\right)\right| < \varepsilon$

$$\left|1 + \frac{1}{\sqrt{m}} - 1 + 1 - \left(1 + \frac{1}{\sqrt{n}}\right)\right|$$
$$\leq \left|1 + \frac{1}{\sqrt{m}} - 1\right| + \left|1 - \left(1 + \frac{1}{\sqrt{n}}\right)\right|$$
$$\leq \left|1 + \frac{1}{\sqrt{m}} - 1\right| + \left|1 + \frac{1}{\sqrt{n}} - 1\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since $1 + \frac{1}{\sqrt{n}} \to 1$, $\exists N$ s.t. $\forall n, m \geq N$
$\left|1 + \frac{1}{\sqrt{n}} - 1\right| < \frac{\varepsilon}{2}$ $\forall n \geq N$

$\{a_n\}, \{b_n\}$ both Cauchy & $a_n \leq b_n$ $\forall n$.
Prove that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

$c_n = b_n - a_n \geq 0$

$c = b - a \geq 0$

cauchy ⇒ convergence, say $a_n \to a$ & $b_n \to b$.

$|a_n - a| < \varepsilon$, $|b_n - b| < \varepsilon$ $\forall n \geq N_a, N_b$ resp.

$b_n - a_n \geq 0 \Rightarrow |b_n - a_n| = b_n - a_n$

we want $|b - a| = b - a$

$$|b - a| = \cancel{|(b - b_n) + (b_n - a_n) + (a_n - a)|}$$
$$\cancel{\leq |b - b_n| + |b_n - a_n| + |a_n - a|}$$