$\{a_n\}$ Cauchy $\Rightarrow |a_n - a_m| < \frac{\varepsilon}{3|2a|}$

$\Rightarrow a_n \to a \Rightarrow |a_n - a| < \varepsilon$

To show $\{a_n^2\}$ is Cauchy, i.e. $|a_n^2 - a_m^2| < \varepsilon$

Take $0<\varepsilon<1$

$$|a_n^2 - a_m^2| = |(a_n - a_m)(a_n + a_m)| = |\underbrace{a_n - a_m}_{<\varepsilon}||a_n + a_m|$$

$$= |a_n - a_m||a_n - a + a + a_m|$$

$$\leq |a_n - a_m|\left(|a_n - a| + |a + a_m|\right)$$

$$= |a_n - a_m||a_n - a| + |a_n - a_m||a + a - a + a_m|$$

$$\leq |a_n - a_m||a_n - a| + |a_n - a_m|\left(|2a| + |a_m - a|\right)$$

$$\leq |a_n - a_m||a_n - a| + |a_n - a_m||2a| + |a_n - a_m||a_m - a|$$

$$\leq \frac{\varepsilon}{3|2a|} \cdot \varepsilon + \frac{\varepsilon}{3|2a|} \cdot |2a| + \frac{\varepsilon}{3|2a|} \cdot \varepsilon$$

$$= \frac{\varepsilon^2}{3|2a|} + \frac{\varepsilon}{3} + \frac{\varepsilon^2}{3|2a|} < \varepsilon$$

If $\{a_n^2\}$ is Cauchy, is $\{a_n\}$?

$\{(-1)^n\} = -1, 1, -1, 1, -1, \ldots$ does not converge

$\{(-1)^{2n}\} = \{1\}$ converges

$\{b_n\}$ all positive, $b_n \to 0$,

$|a_m - a_n| \le b_n \ \forall\, m \ge n$

Prove $\{a_n\}$ is Cauchy., i.e. we want an N s.t. Given $\varepsilon > 0$, $|a_n - a_m| < \varepsilon \ \forall\, m, n \ge N$

$b_n \to 0 \Rightarrow$ $\exists N$ s.t. given $\varepsilon > 0$, $|b_n - 0| < \varepsilon \Rightarrow |b_n| < \varepsilon$

$\Rightarrow b_n < \varepsilon \ \forall\, n \ge N$

So, $|a_m - a_n| \le b_n < \varepsilon \ \forall\, m \ge n \ge N$.

2.4 #10, 11

read 2.5 sup & inf