bounded monotone seq. converges

$\rightarrow a_{n+1} > a_n \quad \forall n$ & $\sqrt{2} \le a_n \le 2$

$a_{n+1} = \sqrt{2+\sqrt{a_n}}$

n=1: $a_2 = \sqrt{2+\sqrt{a_1}} = \sqrt{2+\sqrt{2}} > \sqrt{2} = a_1$

$a_2 > 1$

assume n=k: $a_{k+1} = \sqrt{2+\sqrt{a_k}} > \sqrt{2+\sqrt{a_{k-1}}} = a_k$

$a_{k+1} > a_k$

n=k+1: $a_{k+2} = \sqrt{2+\sqrt{a_{k+1}}} = \sqrt{2+\sqrt{2+\sqrt{k}}} > \sqrt{2+\sqrt{2+\sqrt{a_{k-1}}}}$

$a_{k+2} > \sqrt{2+\sqrt{a_k}} = a_{k+1}$

$a_{k+2} > a_{k+1}$

a_n bounded & monotone seq. of real #'s $\Rightarrow a_n$ converges & hence Cauchy

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= 1\cdot a^n + \frac{n!}{2!(n-2)!} a^{n-1} b + \frac{n!}{3!(n-3)!} a^{n-2} b^2 + \dots 1\cdot b^n$$

$$(1+a_n)^n = 1 + n\cdot a_n + \boxed{\frac{n(n-1)}{2} a_n^2} + \frac{n(n-1)(n-2)}{6} a_n^3 + \dots + a_n^n$$

$\boxed{a_n = \sqrt[n]{n} - 1 \text{. Prove: } n = (1+a_n)^n \ge \frac{n(n-1)}{2} a_n^2}$

$(1+a_n)^n \ge \frac{n(n-1)}{2} a_n^2$ whole is always greater than the part

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$(1+a_n)^n \ge \frac{n(n-1)}{2} a_n^2$ Prove $\sqrt[n]{n} \to 1$

$$\frac{2}{\cancel{n}(n-1)} \cdot \cancel{n} \ge \frac{\cancel{n(n-1)}}{\cancel{2}} (\sqrt[n]{n} - 1)^2 \cdot \cancel{\frac{2}{n(n-1)}}$$

$$(\sqrt[n]{n} - 1)^2 \le \frac{2}{n-1}$$

$$|\sqrt[n]{n} - 1| \le \sqrt{\frac{2}{n-1}}$$