3.9 - Differentials

Recall:
For a function f that is differentiable at c, the equation of the tangent line at the point $(c, f(c))$ is given by

$y - f(c) = f'(c)(x - c)$

This follows from the point-slope equation $y - y_1 = m(x - x_1)$, where the slope m is the derivative $f'(x)$ evaluated at the point $(c, f(c))$.

Since c, $f(c)$, and $f'(c)$ are all constants, if we rearrange to solve for y,

$y = f(c) + f'(c)(x - c)$

y is a linear function of x, called the linear approximation or tangent line approximation to the graph of $f(x)$ at $x = c$.

$T(x) = f(c) + f'(c)(x - c)$

For values of x close to c, values of $y = T(x)$ can be used as approximations of the values of the original function f.

Recall that the slope of the *secant line* through two points $(c, f(c))$ and $(x, f(x))$ is given by $\frac{\Delta y}{\Delta x} = \frac{f(x)-f(c)}{x-c}$, and the slope of the *tangent line* is the limit as the distance between these two points goes to zero of this expression, which we define to be the derivative.

Noting that the change in x is $\Delta x = x - c$, or $x = c + \Delta x$ and hence $f(x) = f(c + \Delta x)$, we can write this two ways:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Actual change in y is $\Delta y = f(x) - f(c) = f(c + \Delta x) - f(c)$.

Recalling the tangent line *approximation* equation

$T(x) = f(c) + f'(c)(x - c) = f(c) + f'(c)\Delta x$

We can see that change in y can be approximated by $T(x) - f(c)$, or

Approximate change in y is $\Delta y \approx f'(c)\Delta x$.

For such an approximation, Δx is denoted dx, and is called the differential of x. The expression $f'(x)dx$ is denoted by dy and called the **differential of y**.

$$\boldsymbol{dy = f'(x)dx}$$

In many applications, the differential of y can be used as an approximation of the actual change in y, i.e. $\Delta y \approx f'(x)dx$

All of the differentiation rules can be written in differential form.

By definition of differentials, we have for functions (of x) u and v:

$du = u'dx$ and $dv = v'dx$

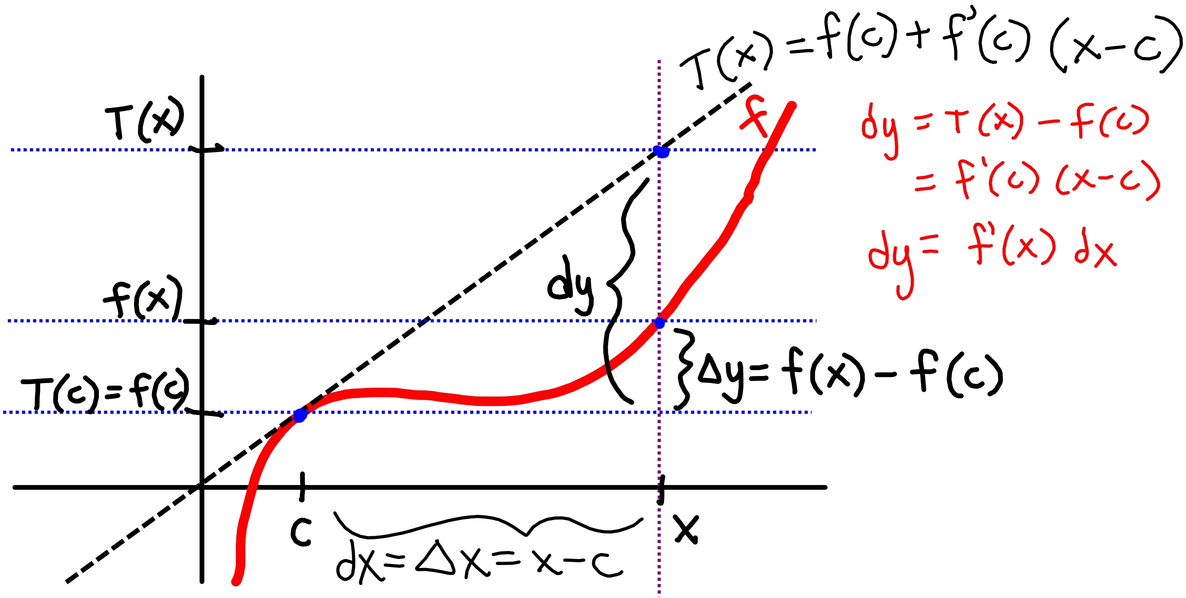
Note that rearranged, these look like $\frac{du}{dx} = u'$ and $\frac{dv}{dx} = v'$.

For example, the Product Rule becomes:

$d[uv] = [uv]'dx = [uv' + vu']dx = uv'dx + vu'dx = udv + vdu$

Differential Formulas

Constant multiple:	$d[cu] = cdu$
Sum or difference:	$d[u \pm v] = du \pm dv$
Product:	$d[uv] = udv + vdu$
Quotient:	$d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$



$dy = T(x) - f(c)$
$\quad = f'(c)(x-c)$

$dy = f'(x)\,dx$

slope of tangent to f @ c is $f'(c)$

eq. of tangent @ c is $y - f(c) = f'(c)(x-c)$

$T(x) = y = f(c) + f'(c)(x-c)$

<u>3.9 #8</u> $y = 1 - 2x^2 = f(x)$; $x = 0$; $\Delta x = dx = -0.1$

Compare dy and Δy for the given values of x and Δx.

$\boldsymbol{\Delta y = f(c + \Delta x) - f(c)}$ $\boldsymbol{dy = f'(x)dx}$

$\Delta y = f(0 - 0.1) - f(0)$

$1 - 2(-0.1)^2$

$1 - 2(0.01)$

0.98

$\Delta y = 0.98 - 1$

$= 0.02$

$dy = (-4x)(-0.1)$

$= 0.4x$

$= 0.4(0)$

$= 0$

$dy = -4x\,dx$

$= -4(0)(-0.1)$

$= 0$