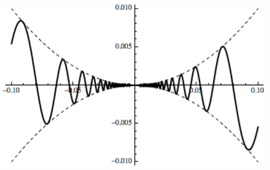
Basic example

The basic example of a differentiable function with discontinuous derivative is

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

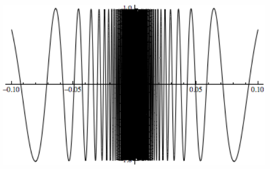
The differentiation rules show that this function is differentiable away from the origin and the difference quotient can be used to show that it is differentiable at the origin with value $f'(0) = 0$. A graph is illuminating as well as it shows how $\pm x^2$ forms an envelope for the function forcing differentiablity.



The derivative of f is

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

which is discontinuous at $x = 0$. It's graph looks something like so



http://math.stackexchange.com/questions/292275/discontinuous-derivative

12. Consider the differentiable function $f(x)$. Some known values of f are given in the table below:

x	2.0	2.2	2.4	2.6
$f(x)$	3	4	6	9

Use the given values to estimate $f'(2.1)$.

(A) 0.5
(B) 3
(C) 3.5
(D) 4
$\boxed{\text{(E) } 5}$

$$f'(x) \approx \frac{f(x) - f(x_1)}{x - x_1}$$

$$\frac{4-3}{2.0-2.2} = \frac{1}{0.2} = \frac{10}{2} = 5$$

13. Consider the differentiable function $y = h(t)$, which represents the height of coffee in a coffee pot, t minutes after the coffee begins to brew. Assume that it takes 5 minutes to brew a pot of coffee, that the pot is initially empty, and that no coffee is taken out of the pot while the coffee is brewing. Which of the following statements must true?

   I. $h'(t) \geq 0 \forall t \in (0,5)$ ✓
   II. ~~$h'(2)$ represents the slope of the secant line of $y = h(t)$ on $0 \leq t \leq 2$~~
   III. $h'(2)$ represents the rate at which the height of coffee in the pot is rising at $t = 2$ minutes. ✓

   (A) I only
   (B) I and II only
   (C) I and III only
   (D) II and III only
   (E) I, II and III

is $\dfrac{h(t_2) - h(t_1)}{t_2 - t_1} \geq 0$

14. Consider the function $Q(t)$, representing the temperature, measured in °C, for a city where t is measured in hours since midnight. If the average rate of change in Q on the interval $[5,10]$ is $2\frac{C}{hr}$ then

   (A) ~~the temperature at 10:00 a.m. was 20°C.~~
   (B) the temperature increased by 10°C between 5:00 a.m. and 10:00 a.m.
   (C) the temperature was increasing at 5:00 a.m.
   (D) $Q'(5) = 2$
   (E) the coldest temperature between midnight and 10:00 a.m. occurred before 5:00 a.m.

avg r.o.c

$\dfrac{\Delta Q}{\Delta t} = \dfrac{10°}{10-5}$

$= 2°C/hr$

15. Which of the following statements are true about $\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}$?

I. ~~$\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}$ represents the slope of the tangent to $f(x)=\sqrt{x+2}$ at any given value of x.~~

II. $\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}=\frac{1}{4}$ ✓

III. $\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}$ represents $f'(2)$ if $f(x)=\sqrt{x+2}$ ✓

(A) I only
(B) I and II only
(C) I and III only
$\boxed{\text{(D) II and III only}}$
(E) I, II and III

$$\lim_{x\to 2}\frac{\sqrt{x+2}-\sqrt{2+2}}{x-2}=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$$

$$f(x)=(x+2)^{1/2}$$

$$f'(x)=\frac{1}{2}(x+2)^{-1/2}=\frac{1}{2\sqrt{x+2}}$$

16. A ball is thrown up in the air from the ground. The height of the ball above the ground at time t is given by the function $h(t)$.

(a) What is the average velocity of the ball over its entire trajectory—i.e., from the moment it is thrown until it hits the ground?

(b) Use the characterization of the derivative as the slope of the tangent line to explain why the instantaneous velocity of the ball must be zero at some point before it hits the ground.

(a)
$$V_{avg}=\frac{\Delta\text{position}}{\Delta\text{time}}=\frac{0}{\Delta\text{time}}=0$$

(b) The Mean Value Theorem says that If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$, i.e. instantaneous rate of change is equal to average rate of change.

Linear motion is continuous, so since $V_{avg}=\frac{h(t_{final})-h(t_{initial})}{t_{total}}=0$, there must be some t for which $h'(t)=0$.

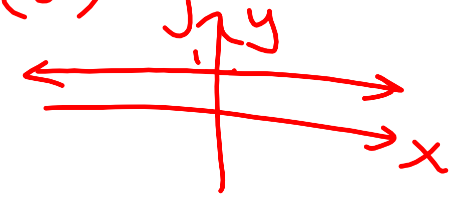
17. For each part of the question, sketch a function that satisfies the given criteria or explain why no function exists that satisfies the given conditions.

    (a) f is continuous and differentiable everywhere.

    (b) f is differentiable everywhere but discontinuous at $x = 1$.

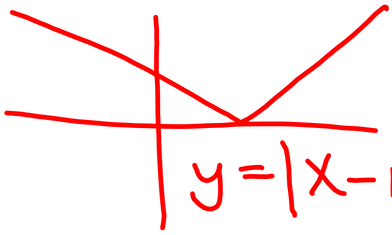
    (c) f is defined everywhere but not differentiable at $x = 1$.

(a) $y = 1$



(b) does not exist since differentiability implies continuity, If f is discontinuous @ $x=1$, f can not be differentiable there.

(c)



$y = |x-1|$

18. Suppose that the derivative of the function $f(x)$ is 1 at the point $x = a$. Use the characterization of the derivative as the slope of the tangent line to explain why $f(b) > f(a)$ for $b > a$ and b sufficiently close to a.

$$f'(a) \approx \frac{f(b)-f(a)}{b-a}$$

for b sufficiently close to a

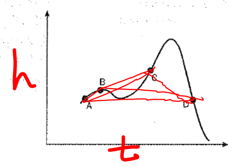
$$\cancel{k}\, 1 \approx \frac{f(b)-f(a)}{b-a}$$

Since $b>a$, $b-a>0$

$f(b)-f(a)$ must be positive in order for $\frac{f(b)-f(a)}{b-a} \approx 1 > 0$,

so $f(b) > f(a)$.

19. The height of a particle above the ground at time t is given by the graph below.



(a) Between which two labeled points on the graph is the particle's average velocity the least? $\frac{\Delta h}{\Delta t}$

(b) At which of the labeled points on the graph is the particle's instantaneous velocity the greatest? $h'(t)$

(c) At which of the labeled points on the graph is the particle changing directions?

(d) At which of the labeled points on the graph is the particle's instantaneous velocity zero?

(a) The slope of the line through points C&D is the most negative.

(b) slope of tangent line at point C is greatest

(c) The height of the particle is increasing to the left of B and decreasing to the right.

(d) The slope of the tangent line @ point B is zero

1. If $f(x)=x^5-2x^2+\frac{3}{x}$, then $f'(-1)=$

(A) -11
(B) -2
(C) 2
(D) 6
(E) 11

$f'(x)=5x^4-4x-\frac{3}{x^2}=5+4-3$

$\left(\frac{1}{x}\right)'=\frac{-1}{x^2}$ $\quad (\sqrt{x})'=\frac{1}{2\sqrt{x}}$

2. A particle is traveling along the x-axis. Its position is given by $x(t)=\frac{1-t^2}{t+3}$ at time $t\geq 0$. Find the instantaneous rate of change of x with respect to t when $t=1$.

(A) -2
(B) $-\frac{1}{2}$
(C) 0
(D) $\frac{1}{2}$
(E) 2

$x'(t)=$

$\left(\frac{f}{g}\right)'=$ "low dee high less high dee low draw the line and square below"

$$x'(t)=\frac{(t+3)(-2t)-(1-t^2)(1)}{(t+3)^2}=\frac{g\cdot f'-f\cdot g'}{g^2}$$

$$x'(1)=\frac{4(-2)-0}{16}=-\frac{1}{2}$$

3. If $f(x) = \cos(\ln x)$ for $x > 0$, then $f'(x) =$

(A) $-\sin(\ln x)$

(B) $\sin(\ln x)$

(C) $\boxed{-\frac{\sin(\ln x)}{x}}$

(D) $\frac{\sin(\ln x)}{x}$

(E) $\sin\left(\frac{\ln x}{x}\right)$

$$-\sin(\ln x)\cdot\frac{1}{x}$$

4. If $f(x) = x \cdot 2^x$, then $f'(x) =$

(A) $2^x(x + \ln 2)$

(B) $2^x(1 + \ln 2)$

(C) $x \cdot 2^x \cdot \ln 2$

(D) $2^x(1 + x \cdot \ln 2)$

(E) $x \cdot 2^x(1 + \ln 2)$

$$x \cdot 2^x \ln 2 + 1 \cdot 2^x$$

$$2^x(x\ln 2 + 1)$$

$f'(x) = 3x^2 - 1$

5. Let $f(x) = x^3 - x + 2$. If h is the inverse of f, then $h'(2)$ could be

(A) $\frac{1}{26}$

(B) $\frac{1}{4}$

(C) $\boxed{\frac{1}{2}}$

(D) 2

(E) 26

If $f(a) = b$, $[f^{-1}(b)]' = \dfrac{1}{f'[f^{-1}(b)]} = \dfrac{1}{f'(a)}$

$f(x) = 2$, $[f^{-1}(2)]' = \dfrac{1}{f'(x)}$

$x^3 - x + \cancel{2} = \cancel{2}$

$x(x^2 - 1) = 0$

$x = 0, 1, -1$

$f'(x) = 3x^2 - 1$

$f'(0) = -1$, $f'(1) = 2$, $f'(-1) = 2$

$\frac{1}{-1}$, $\frac{1}{2}$, $\frac{1}{2}$

6. Let $f(x) = x \cdot g(h(x))$, where $g(4) = 2, g'(4) = 3, h(3) = 4$, and $h'(3) = -2$. Find $f'(3)$.

(A) -18

(B) -16

(C) -7

(D) 7

(E) 11