$\{a_n\}$ converges to the limit a if given $\varepsilon > 0$, there exists $N > 0$ $(N(\varepsilon) > 0)$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

$\{a_n\}$ diverges to $\pm\infty$ if given $M \in \mathbb{R}$, there exists $N > 0$ such that $|a_n| > M$ $\forall$ $n \geq N$

"for all"

2, 4, 6, 8, 10, 12, . . .

2.1.2 b

$a_n = 1 + \frac{1}{\sqrt[3]{n}}$

Claim: $a_n \to 1$.

Let $\varepsilon > 0$ be given. We want to find an $N > 0$ in terms of $\varepsilon > 0$, such that $|a_n - a| < \varepsilon \;\; \forall\, n \geq N$.

$|a_n - a| = \left|1 + \frac{1}{\sqrt[3]{n}} - 1\right| = \left|\frac{1}{\sqrt[3]{n}}\right| < \varepsilon \Rightarrow \frac{1}{\sqrt[3]{n}} < \varepsilon$

$\Rightarrow \sqrt[3]{n} > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon^3}$

Take $N = \frac{1}{\varepsilon^3}$. Then, whenever $n \geq N$, we have that $|a_n - a| = \left|\frac{1}{\sqrt[3]{n}}\right| = \frac{1}{\sqrt[3]{n}} < \frac{1}{\sqrt[3]{1/\varepsilon^3}} = \varepsilon$, i.e. $|a_n - a| < \varepsilon$, and hence $a_n \to 1$. $\square$

2.1.4 b

$a_n = -n^2$

Let $M \in \mathbb{R}$ be given. $(M < 0)$.

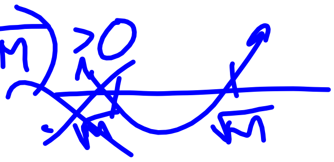
We want $N > 0$ st. $a_n < M \;\; \forall\, n \geq N$.

$-n^2 < M \Rightarrow n^2 > -M \Rightarrow n > \sqrt{-M}$

Take $N = \sqrt{-M}$.

$n^2 + M > 0$

$(n - \sqrt{M})(n + \sqrt{M}) > 0$



2.1.5 Limits are unique.

Proof by contradiction

Suppose $\{a_n\}$ converges to more than one limit, say $a_n \to a$ and $a_n \to b$.

$a_n \to a \Rightarrow$ Given $\varepsilon > 0$, $\exists N_1 > 0$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$ $\forall n \geq N_1$.

$a_n \to b \Rightarrow$ Given $\varepsilon > 0$, $\exists N_2 > 0$ s.t. $|a_n - b| < \frac{\varepsilon}{2}$ $\forall n \geq N_2$.

Let $\varepsilon > 0$ be given. Take $N = \max\{N_1, N_2\}$.

$|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow a = b$.

2.1

\# 2 a, c, d

3 a, b, c

4 a, c, d

6

7