MA 061 Geometry I – Chapters 2-10 Definitions, Postulates, Theorems, Corollaries, and Formulas

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Chapter 2 – The Nature of Deductive Reasoning

conditional statement: "If a, then b." or "a implies b." or $a \rightarrow b$

a = **hypothesis**
b = **conclusion**

converse: "If b, then a." or "b implies a." or $b \rightarrow a$
In general, if a statement is true, then its converse is not necessarily true.

contrapositive: "If not b, then not a." or "not b implies not a." or $\sim b \rightarrow \sim a$
The contrapositive is logically equivalent to the original statement.

If both a statement and its converse are true, we can combine them with an **if and only if** (iff) statement: "a if and only if b" or $a \leftrightarrow b$
Statements of definitions are always true, as are their converses.

Euler diagram for $a \rightarrow b$ (b (a))

A **syllogism** is a type of **direct proof** of the form
a →b
b→c
Therefore, a→c.

The statements a→b and b→c are called the **premises** of the argument.
a→c is called the **conclusion** of the argument, and is often considered to be a theorem.

A **theorem** is a statement that is proved by reasoning deductively from already accepted statements.

If the premises of a syllogism are true, it follows that its conclusion must be true.
If the premises of a syllogism are false, the conclusion may be true or false.

In an **indirect proof**, an assumption is made at the beginning that leads to a contradiction. The contradiction indicates that the assumption is false and the desired conclusion is true.

Direct versus Indirect proof of the theorem "If a, then d."

Direct Proof:
If a, then b.
If b, then c.
If c, then d.
Therefore, if a, then d.

Indirect Proof:
Suppose not d is true.
If not d, then e.
If e, then f,
And so on until we come to a contradiction.
Therefore, our assumption (not d) is false; so d is true.

To avoid circular definitions, mathematics leaves certain terms **undefined** (e.g. point, line, plane), which can be used to define other terms.

Def: Points are **collinear** iff there is a line that contains all of them.
Def: Lines are **concurrent** iff they contain the same point.

Def: A **postulate** is a statement that is assumed to be true without proof.

Postulate 1: **Two points determine a line.**
Postulate 2: **Three noncollinear points determine a plane.**

The Pythagorean Theorem: The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

The Triangle Sum Theorem: The sum of the angles in a triangle is 180°.

Circle Theorems:
If the diameter of a circle is d, then its circumference is πd.
If the radius of a circle is r, then its area is πr^2.

Chapter 3 – Lines and Angles

Algebraic Postulates of Equality:
Reflexive Property: a=a (Any number is equal to itself.)
Substitution Property: If a=b, then a can be substituted for b in any expression.
Addition Property: If a=b, then a+c=b+c
Subtraction Property: If a=b, then a-c=b-c.
Multiplication Property: If a=b, then ac=bc.
Division Property: If a=b, then a/c=b/c.

Quadratic formula If $ax^2 + bx + c = 0$, then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Postulate 3: **The Ruler Postulate** – The points on a line can be numbered so that positive number differences measure distance.

Def: **Betweenness of Points** – A point is between two other points on the same line iff its coordinate is between their coordinates. (More briefly, **A-B-C iff a<b<c or a>b>c**.)

Theorem 1: **The Betweenness of Points Theorem – If A-B-C, then AB+BC=AC**

Postulate 4: **The Protractor Postulate** – The rays in a half-rotation can be numbered from 0 to 180 so that positive number differences measure angles.

Definitions: An angle is
Acute iff it is less than 90°.
Right iff it is 90°.
Obtuse iff it is more than 90° but less than 180°.
Straight iff it is 180°.

Def: **Betweenness of Rays** – A ray is between two others in the same half-rotation iff its coordinate is between their coordinates. (More briefly, **OA-OB-OC iff a<b<c or a>b>c**.)

Theorem 2: **The Betweenness of Rays Theorem – If OA-OB-OC, then ∠AOB+∠BOC=∠AOC.**

Def: A point is on the **midpoint of a line segment** iff it divides the line segment into two equal segments.

Def: A line **bisects an angle** iff it divides the angle into two equal angles.

Def: Two objects are **congruent** if and only if they coincide exactly when superimposed.

Def: A **corollary** is a theorem that can be easily proved as a consequence of a postulate or another theorem.

Corollary to the Ruler Postulate: **A line segment has exactly one midpoint.**

Corollary to the Protractor Postulate: **An angle has exactly one ray that bisects it.**

Def: Two angles are **complementary** iff their sum is 90°.

Def: Two angles are **supplementary** iff their sum is 180°.

Theorem 3: **Complements of the same angle are equal.**

Theorem 4: **Supplements of the same angle are equal.**

Def: Two angles are a **linear pair** iff they have a common side and their other sides are opposite rays.

Def: Two angles are **vertical angles** iff the sides of one angle are opposite rays to the sides of the other.

Theorem 5: **The angles in a linear pair are supplementary.**

Theorem 6: **Vertical angles are equal.**

Def: Two lines are **perpendicular** iff they form a right angle.

Theorem 7: **Perpendicular lines form four right angles**.

Corollary to the definition of a right angle: **All right angles are equal**.

Theorem 8: **If the angles in a linear pair are equal, then their sides are perpendicular**.

Def: Two lines are **parallel** iff they lie in the same plane and do not intersect.

Chapter 4 – Congruence

Distance formula: The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Def: A **polygon** is a connected set of at least three line segments in the same plane such that each segment intersects exactly two others, one at each endpoint. The line segments are the sides of the polygon, and the endpoints are its vertices. The number of sides and vertices is always the same, and the polygon is referred to as an "n-gon" if it has n sides and n vertices.

Def: Two triangles are **congruent** iff there is a correspondence between their vertices such that all of their corresponding sides and angles are equal. $\Delta ABC \cong \Delta LMN$, where "$\cong$" means "is congruent to" $ABC \leftrightarrow LMN$

Corollary to the definition of congruent triangles: **Two triangles congruent to a third triangle are congruent to each other.**

A polygon is **convex** if any line connecting two points of the polygon never goes outside the polygon.
A polygon is **concave** if there exists at least one set of points of the polygon such that a line connecting them must travel outside the polygon.

Postulate 5: The **ASA** Postulate
If two angles and the included side of one triangle are equal to two angles and the included side of another triangle, the triangles are congruent.

Postulate 6: The **SAS** Postulate
If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are congruent.

Def: **Corresponding** parts of **congruent** triangles are equal.

Definitions: A triangle is
scalene iff it has no equal sides
isosceles iff it has at least two equal sides
equilateral iff all of its sides are equal
obtuse iff it has an obtuse angle
right iff it has a right angle
acute iff all of its angles are acute
equiangular iff all of its angles are equal

Theorem 9: **If two sides of a triangle are equal, the angles opposite them are equal.**
Theorem 10: **If two angles of a triangle are equal, the sides opposite them are equal**.
Corollaries to Theorems 9 and 10:
An equilateral triangle is equiangular.
An equiangular triangle is equilateral.

Theorem 11: The **SSS** Theorem
If the three sides of one triangle are equal to the three sides of another triangle, then triangles are congruent.

Chapter 5 – Inequalities

The "Three Possibilities" Property: either a>b, a=b, or a<b
The Transitive Property of Inequality: If a>b and b>c, then a>c
The Addition Property of Inequality: If a>b, then a+c>b+c
The Subtraction Property of Inequality: If a>b, then a-c>b-c
The Multiplication Property of Inequality: If a>b and c>0, then ac>bc
The Division Property of Inequality: If a>b and c>0, then a/c>b/c

The Addition Theorem of Inequality: If a>b and c>d, then a+c>b+d

The "Whole Greater than Part" Theorem: If a>0, b>0, and a+b=c, then c>a and c>b

Def: An **exterior angle** of a triangle is an angle that forms a linear pair with an angle of the triangle.

Theorem 12: The Exterior Angle Theorem – An Exterior angle of a triangle is greater than either remote interior angle.

Theorem 13: If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.

Theorem 14: If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.

Theorem 15: The Triangle Inequality Theorem – The sum of any two sides of a triangle is greater than the third side.

Chapter 6 – Parallel Lines

Def: Two points are **symmetric with respect to a line** iff the line is the perpendicular bisector of the line segment connecting the two points.

Theorem 16: In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

Def: Two lines are **parallel** iff they lie in the same plane and do not intersect.

A **transversal** is a line that intersects two or more lines in different points.

Theorem 17: Equal corresponding angles mean that lines are parallel.
Corollary 1: Equal alternate interior angles mean that lines are parallel.
Corollary 2: Supplementary interior angles on the same side of a transversal mean that lines are parallel.
Corollary 3: In a plane, two lines perpendicular to a third line are parallel.

Postulate 7: The Parallel Postulate – Through a point not on a line, there is exactly one line parallel to the given line.

Theorem 18: In a plane, two lines parallel to a third line are parallel to each other.

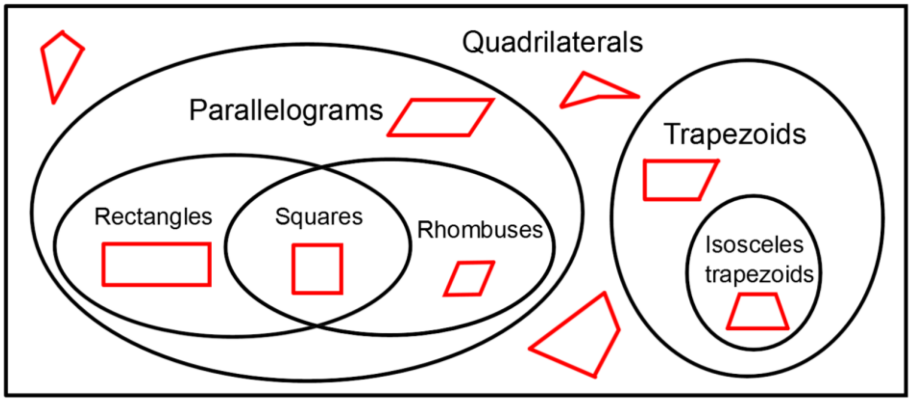
Theorem 19: Parallel lines form equal corresponding angles.
Corollary 1: Parallel lines form equal alternate interior angles.
Corollary 2: Parallel lines form supplementary interior angles on the same side of a transversal.
Corollary 3: In a plane, a line perpendicular to one of two parallel lines is also perpendicular to the other.

Theorem 20: The Angle Sum Theorem – The sum of the angles of a triangle is 180°.
Corollary 1: If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.
Corollary 2: The acute angles of a right triangle are complementary.
Corollary 3: Each angle of an equilateral triangle is 60°.

Theorem 21: An exterior angle of a triangle is equal to the sum of the remote interior angles.

Theorem 22: The **AAS** Theorem – If two angles and the side opposite one of them in one triangle are equal to the corresponding parts of another triangle, the triangles are congruent.

Theorem 23: The **HL** Theorem – If the hypotenuse and a leg of one right triangle are equal to the corresponding parts of another right triangle, the triangles are congruent.



Chapter 7 - Quadrilaterals

Def: A **diagonal** of a polygon is a line segment that connects any two nonconsecutive vertices.

Theorem 24: The sum of the angles of a quadrilateral is 360°.

Def: A **rectangle** is a quadrilateral each of whose angles is a right angle.
Corollary to Theorem 24: A quadrilateral is equiangular iff it is a rectangle.

Def: A **parallelogram** is a quadrilateral whose opposite sides are parallel.

A figure has point symmetry if it looks exactly the same when it is rotated about a point.
Def: Two points are **symmetric with respect to a point** iff it is the midpoint of the line segment joining them.
Parallelograms have point symmetry about the point in which their diagonals intersect.

Theorem 25: The opposite sides and angles of a parallelogram are equal.

Theorem 26: The diagonals of a parallelogram bisect each other.

Theorem 27: A quadrilateral is a parallelogram, if its opposite sides are equal.

Theorem 28: A quadrilateral is a parallelogram if its opposite angles are equal.

Theorem 29: A quadrilateral is a parallelogram if two opposite sides are both parallel and equal.

Theorem 30: A quadrilateral is a parallelogram if its diagonals bisect each other.

Def: A **square** is a quadrilateral all of whose sides and angles are equal.

Every square is a rhombus.

Def: A **rhombus** is a quadrilateral all of whose sides are equal.

Theorem 31: All rectangles are parallelograms.
Given: ABCD is a rectangle.
Prove: ABCD is a parallelogram.

Theorem 32: All rhombuses are parallelograms.
Given: ABCD is a rhombus.
Prove: ABCD is a parallelogram.

Theorem 33: The diagonals of a rectangle are equal.
Given: ABCD is a rectangle.
Prove: AC=BD.

Theorem 34: The diagonals of a rhombus are perpendicular.
Given: ABCD is a rhombus.
Prove: AC⊥BD.

Def: A **trapezoid** is a quadrilateral that has exactly one pair of parallel sides.
The parallel sides are called the bases of the trapezoid, and the non-parallel sides are called its legs. The pairs of angles that include each base are called base angles.

Def: An **isosceles trapezoid** is a trapezoid whose legs are equal.

Theorem 35: The base angles of an isosceles trapezoid are equal.
Given: ABCD is an isosceles trapezoid with bases AB and DC.
Prove: ∠A=∠B and ∠D=∠C

Theorem 36: The diagonals of an isosceles trapezoid are equal.
Given: ABCD is an isosceles trapezoid with bases AB and DC.
Prove: DB=CA.

If a quadrilateral is a trapezoid, then its diagonals cannot bisect each other.
Given: ABCD is a trapezoid
Prove: AC and DB do not bisect each other.

Def: A **midsegment** of a triangle is a line segment that connects the midpoints of two of its sides.

Theorem 37: The Midsegment Theorem – A midsegment of a triangle is parallel to the third side and half as long.
Given: MN is a midsegment of ΔABC.
Prove: MN||BC and MN=(1/2)BC.

Chapter 8 – Transformations

Def: A transformation is a one-to-one correspondence between two sets of points.

A translation slides an object a certain distance without turning it.

A reflection flips an object over a mirror line.

A rotation turns an object a certain number of degrees about a fixed point.

A dilation enlarges or reduces the size of an object.

Def: an isometry is a transformation that preserves distance and angle measure.

Translations, reflections, and rotations are all examples of isometries, but dilations are not.

Def: The reflection of point P through line l is P itself if P lines on l. Otherwise, it is the point P' such that l is the perpendicular bisector of PP'.

Def: A translation is the composite of two successive reflections through parallel lines.

The distance between a point of the original figure and its translation image is called the *magnitude* of the translation.

Def: A rotation is the composite of two successive reflections through intersecting lines.

The point in which the lines intersect is the *center* of rotation, and the measure of the angle through which a point of the original figure turns to coincide with its rotation image is called the *magnitude* of the rotation.

Def: A translation is the composite of two successive reflections through parallel lines.
The distance between a point of the original figure and its translation image is called the *magnitude of the translation.*

Def: A rotation is the composite of two successive reflections through intersecting lines.
The point in which the lines intersect is the *center of rotation*, and the measure of the angle through which a point of the original figure turns to coincide with its rotation image is called the *magnitude of the rotation*.

Def: Two figures are congruent if there is an isometry such that one figure is the image of the other.

Def: A glide reflection is the composite of a translation and a reflection in a line parallel to the direction of the translation.

Def: A figure has rotation symmetry with respect to a point iff it coincides with its rotation image through less than 360° about the point.

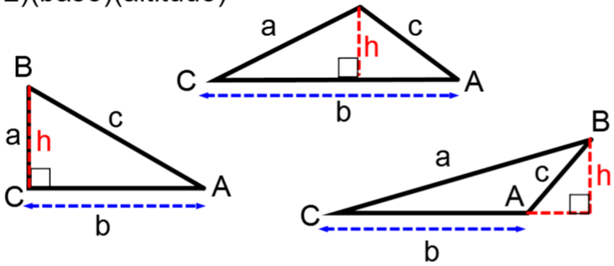
A figure is said to have n-fold rotation symmetry iff the smallest angle through which it can be turned to look exactly the same is 360°/n.

Def: A figure has reflection (line) symmetry with respect to a line iff it coincides with its reflection image through the line. The line is sometimes called the axis of symmetry.

Def: A pattern has translation symmetry iff it coincides with a translation image.

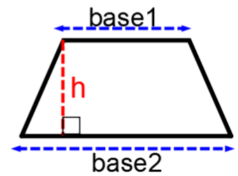
Area Review

Area of any triangle with known altitude is (1/2)(base)(altitude)

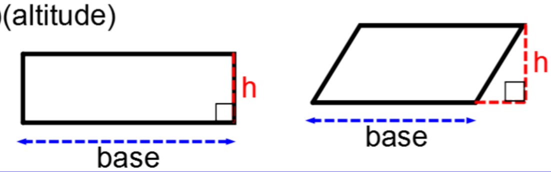


Area of any triangle with unknown altitude, but known side lengths a, b, and c, is $\sqrt{s(s-a)(s-b)(s-c)}$ where s is half the perimeter s=(1/2)(a+b+c)

Area of any trapezoid is (1/2)(base1 + base2)(altitude)



Area of any parallelogram (including rectangles) is (base)(altitude)



Chapter 9 – Area

<u>Postulate 8</u> - **<u>The Area Postulate</u>**
Every polygonal region has a positive number called its area such that
(1) congruent triangles have equal areas
(2) the area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts

<u>Postulate 9</u> - The area of a rectangle is the product of its base and altitude
<u>Corollary to Postulate 9</u> - The area of a square is the square of its side

<u>Theorem 38</u> - The area of a right triangle is half the product of its legs.

<u>Theorem 39</u> - The **<u>area of a triangle</u>** is half the product of any base and corresponding altitude.
<u>Corollary to Theorem 39</u> - Triangles with equal bases and equal altitudes have equal areas.

<u>Heron's Formula</u> – The area of a triangle with sides a, b, and c is $\sqrt{s(s-a)(s-b)(s-c)}$, where s is half the triangle's perimeter $s = \frac{a+b+c}{2}$

<u>Theorem 40</u> - The **<u>area of a parallelogram</u>** is the product of any base and corresponding altitude.

<u>Theorem 41</u> - The **<u>area of a trapezoid</u>** is half the product of its altitude and the sum of its bases.

<u>Theorem 42 (**The Pythagorean Theorem**)</u> - The square of the hypotenuse of a right triangle is equal to the sum of the squares of its legs.
<u>Theorem 43 (Converse of the Pythagorean Theorem)</u> - If the square of one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle.

Chapter 10 - Similarity

Def: The **ratio** of the number a to the number b is the number $\frac{a}{b}$.

A **proportion** is an equality between ratios. $\frac{a}{b} = \frac{c}{d}$

a, b, c, and d are called the *first, second, third, and fourth terms*.

The second and third terms, b and c, are called the means.

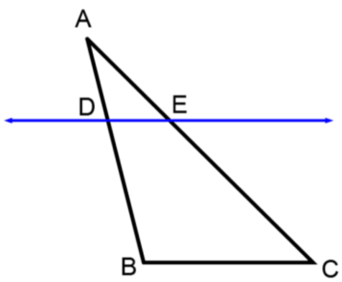
The first and fourth terms, a and d, are called the extremes.

The product of the means is equal to the product of the extremes.

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

Def: The number b is the **geometric mean** between the numbers a and c if a, b, and c are positive and $\frac{a}{b} = \frac{b}{c}$.

Def: Two triangles are **similar** iff there is a correspondence between their vertices such that their corresponding sides are proportional and their corresponding angles are equal.



Theorem 44 - **The Side-Splitter Theorem**

If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides in the same ratio, that is, if in triangle ABC, DE||BC, then $\frac{AD}{DB} = \frac{AE}{EC}$.

Corollary to the Side-Splitter Theorem:

If a line parallel to one side of a triangle intersects the other two sides in different points, it cuts off segments proprtional to the sides, that is, $\frac{AD}{AB} = \frac{AE}{AC}$ and $\frac{DB}{AB} = \frac{EC}{AC}$

Theorem 45 - **The AA Theorem** - If two angles of one triangle are equal to two angles of another triangle, the triangles are similar.

Corollary to the AA Theorem - Two triangles similar to a third triangle are similar to each other.

Theorem 46 - Corresponding altitudes of similar triangles have the same ratio as that of the corresponding sides.

Given: $\Delta ABC \sim \Delta DEF$; BG and EH are altitudes Prove: $\frac{BG}{EH} = \frac{AC}{DF}$

Theorem 47 – The ratio of the perimeters of two similar polygons is equal to the ratio of the corresponding sides.

Given: $\Delta ABC \sim \Delta A'B'C'$ Prove: $\frac{\rho \Delta ABC}{\rho \Delta A'B'C'} = r$, where $r = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$

Theorem 48 – The ratio of the areas of two similar polygons is equal to the square of the ratio of the corresponding sides.

Given: $\Delta ABC \sim \Delta A'B'C'$ Prove: $\frac{\propto \Delta ABC}{\propto \Delta A'B'C'} = r^2$, where $r = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$

SAS Similarity Theorem: If an angle of one triangle is equal to an angle of another triangle and the sides including these angles are proportional, then the triangles are similar.

Given: ΔABC and $\Delta A'B'C'$ with $\angle A = \angle A'$ and $\frac{b}{b'} = \frac{c}{c'}$.

Prove: $\Delta ABC \sim \Delta A'B'C'$

SSS Similarity Theorem: If the sides of one triangle are proportional to the sides of another triangle, then the triangles are similar.

Given: ΔABC and $\Delta A'B'C'$ with $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Prove: $\Delta ABC \sim \Delta A'B'C'$