Precalculus Notes

Linear Functions

A linear function is one of the form $f(x) = mx + b$, where m is the slope of the line and b is the y-intercept. $y = mx + b$ is called the slope-intercept form of the equation of a line.
The slope of a linear function can be found by taking the ratio of change in y-values over the change in x-values.
$m = \frac{y_2 - y_1}{x_2 - x_1} = "\frac{rise}{run}"$
Given the slope m and a point (x_1, y_1) on a line, the slop-intercept form can be easily found by plugging these values into the point-slope equation: $y - y_1 = m(x - x_1)$.
Lines with a 0-slope are called horizontal lines and are of the form $y = k$ for some constant k. Vertical lines are said to have "no slope" and are of the form $x = k$.
Two lines in a plane are parallel if they never intersect. Two lines are perpendicular if their intersection forms a 90° angle.
Let l_1 be the graph of $f_1(x) = m_1x + b_1$ and let l_2 be the graph of $f_2(x) = m_2x + b_2$. l_1 and l_2 are parallel if $m_1 = m_2$. This is denoted $l_1 \parallel l_2$. l_1 and l_2 are perpendicular if $m_1 = -\frac{1}{m_2}$. This is denoted $l_1 \perp l_2$.

Complex Numbers

Standard form: $a + bi$, where a is the real part and b is the imaginary part
$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$
$i^a = i^{4b+r} = (i^4)^b i^r = i^r, where\ r\ is\ the\ remainder\ (0, 1, 2\ or\ 3)\ when\ a\ is\ divided\ by\ 4$
$a + bi$ and $a - bi$ are complex conjugates; their product $(a + bi)(a - bi)$ is the real number $a^2 + b^2$
To add complex numbers $a + bi$ and $c + di$, just combine like terms: $(a + c) + (b + d)i$
To multiply complex numbers $(a + bi)(c + di)$, FOIL, combine like terms, and replace i^2 with -1.
To divide complex numbers, multiply numerator and denominator by the conjugate of the denominator.

$$\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}$$

Caution: to multiply $\sqrt{a}\sqrt{b}$ when $a, b < 0$, write in terms of i before multiplying.

Quadratic Equations

Standard form: $ax^2 + bx + c = 0, \ \ a \neq 0$
Zero Product Property: $If\ AB = 0,\ then\ A = 0\ or\ B = 0$
Square Root Theorem: $If\ A^2 = B,\ then\ A = \pm\sqrt{B}$
Quadratic Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The discriminant is the $b^2 - 4ac$ part of the quadratic formula.
If $b^2 - 4ac > 0$, the quadratic equation will have two distinct real roots (solutions).
If $b^2 - 4ac = 0$, the quadratic equation will have one real "double" root.
If $b^2 - 4ac < 0$, the quadratic equation will have two complex conjugate roots.

Steps for Completing the Square:
1. Move the constant term to the right-hand side.

$$ax^2 + bx = -c$$

2. Factor out the x^2 coefficient from all terms on the left-hand side.

$$a(x^2 + \frac{b}{a}x) = -c$$

3. Divide both sides by the x^2 coefficient.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

4. Complete the square by taking half of the x coefficient, squaring it, and adding it to both sides.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

5. Rewrite the left-hand side as a perfect square and simplify the right-hand side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac+b^2}{4a^2}$$ (Note: this may not look "simplified" but with actual numbers, it will; just write as single fraction)

6. Apply square root theorem.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac+b^2}{4a^2}}$$

7. Solve for x and simplify.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Quadratic Functions

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$. The graph of $f(x)$ is a parabola.
The vertex of a parabola is the lowest point on a parabola that opens up or the highest point on a parabola that opens down.
A parabola is symmetric with respect to the vertical line through its vertex. This line is called the axis of symmetry.
A more useful form of a quadratic function is $f(x) = a(x-h)^2 + k$. The graph of f is a parabola with vertex (h, k).
The parabola opens up if $a >$0 and down if $a < 0$. The vertical line $x = h$ is the axis of symmetry.
The useful form can be gotten from the standard form by completing the square:
Starting with $f(x) = ax^2 + bx + c$,

1. Factor the x^2-coefficient out of the x^2 and x terms only.

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

2. Complete the square inside the parentheses.

$$f(x) = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c$$

3. Add the appropriate constant outside to cancel out the constant we added inside. Adding $\left(\frac{b}{2a}\right)^2$ inside the function is really adding $a \cdot \left(\frac{b}{2a}\right)^2$, since everything inside parentheses is being multiplied by a. Since the only thing we can add to one side of an equation without changing it is 0, we add $-a \cdot \left(\frac{b}{2a}\right)^2 = -\frac{b^2}{4a}$.

$$f(x) = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - \frac{b^2}{4a}$$

4. Rewrite the parentheses as a perfect square and combine constants on the outside into a single integer or fraction.

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Difference Quotient

$$\frac{f(x+h) - f(x)}{h}, h \neq 0$$

Symmetry

The graph of an equation is symmetric with respect to the y-axis if replacement of x with $-x$ leaves the equation unchanged.
The graph of an equation is symmetric with respect to the x-axis if replacement of y with $-y$ leaves the equation unchanged.
The graph of an equation is symmetric with respect to the origin if replacement of x and y with $-x$ and $-y$ leaves the equation unchanged.

Even and Odd Functions

A function f is even if $f(-x) = f(x)$. Even functions are symmetric with respect to the y-axis.
A function f is odd if $f(-x) = -f(x)$. Odd functions are symmetric with respect to the origin.

Polynomials

monomial – a constant, variable, or product of a constant and one or more variables with nonnegative integer exponents
coefficient – the constant part of a monomial
degree of a monomial – the sum of the exponents of the variables
polynomial – sum of a finite number of monomials; each monomial is a term of the polynomial
degree of a polynomial – largest degree of the terms in the polynomial
binomial – simplified polynomial with 2 terms
trinomial – simplified polynomial with 3 terms
general form of an nth degree polynomial is $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0x$
$a_n x^n$ is the lead term
a_n is the leading coefficient
a_0 is the constant term

Factorization theorem: The trinomial $ax^2 + bx + c$, with integer coefficients a, b, c, can be factored as the product of two binomials with integer coefficients if and only if $b^2 - 4ac$ (the discriminant) is a perfect square
Special forms:
$x^2 + 2xy + y^2 = (x + y)(x + y)$
$x^2 - 2xy + y^2 = (x - y)(x - y)$
$x^2 - y^2 = (x - y)(x + y)$
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Other material not included here

Polynomial Division
Theorems about zeros of Polynomials
Rational Functions
Polynomial and Rational Inequalities
Variation

Inverse Functions

A function is one-to-one if $f(a) = f(b)$ implies that $a = b$ for all a, b in the domain of f. That is, in addition to being a function (each x maps to exactly one y), a one-to-one function only has one x-value mapping to each y-value.
The graph of a one-to-one function passes both the horizontal and vertical line tests.

Example of verifying that a function is one-to-one:

$f(x) = (x+4)^3 - 5$

$f(a) = f(b)$

$(a+4)^3 - 5 = (b+4)^3 - 5$

$(a+4)^3 = (b+4)^3$

$\sqrt[3]{(a+4)^3} = \sqrt[3]{(b+4)^3}$

$(a + 4) = (b + 4)$

$a = b$

Since $f(a) = f(b)$ implies that $a = b$, f is one-to-one.

Example showing that a function is NOT one-to-one:

$f(x) = x^2$

$f(2) = 4$

$f(-2) = 4$

Since two different x-values yield the same y-value, f is not one-to-one.

The one-to-one functions $f(x)$ and $g(x)$ are inverses if:

$(f \circ g)(x) = x \ and \ (g \circ f)(x) = x$

Example of verifying that two functions are inverses:

$f(x) = x^3 \quad g(x) = \sqrt[3]{x}$

$(f \circ g)(x) = \left(\sqrt[3]{x}\right)^3 = x$

$(g \circ f)(x) = \sqrt[3]{(x^3)} = x$

How to find an inverse function:

1. Replace $f(x)$ with y
2. Interchange x and y
3. Solve for y in terms of x.
4. Replace y with $f^{-1}(x)$.

Example of finding an inverse function:

$f(x) = \dfrac{5x-3}{2x+1}$

$y = \dfrac{5x-3}{2x+1}$

$x = \dfrac{5y-3}{2y+1}$

$x(2y+1) = 5y - 3$

$2xy + x = 5y - 3$

$x + 3 = 5y - 2xy$

$x + 3 = y(5 - 2x)$

$y = \dfrac{3+x}{5-2x}$

$\boxed{f^{-1}(x) = \dfrac{3+x}{5-2x}}$

Exponential and Logarithmic Functions

Properties of Exponential Functions:

$a^m a^n = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$

$(a^m)^n = a^{mn}$

$(a^p b^q)^r = a^{pr} b^{qr}$, $\left(\frac{a^p}{b^q}\right)^r = \frac{a^{pr}}{b^{qr}}$

$a^{-1} = \frac{1}{a}$

$a^0 = 1, a \neq 0$

Properties of Logarithmic Functions:

Product Rule: $log_a(MN) = log_a M + log_a N$

Power Rule: $log_a(M)^P = p\, log_a M$

Quotient Rule: $log_a\left(\frac{M}{N}\right) = log_a M - log_a N$

Change of Base Formula: $log_b M = \frac{log_a M}{log_a b}$

Other Properties: $log_a a = 1$ $\quad log_a 1 = 0$

$log_a a^x = x$ $\quad a^{log_a x} = x$

$\ln x = \log_e x$ $\quad \log x = \log_{10} x$

The logarithmic equation $\log_a b = c$ is equivalent to the exponential equation $a^c = b$

Compound Interest

The amount of money A that a principal P will grow to after t years at interest rate r (in decimal form), compounded n times per year, is given by the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Exponential Growth and Decay

$P(t) = P_0 e^{kt}$

$P(t)$ = population amount at time t

P_0 = initial population

k = exponential growth/decay rate (K is positive for growth and negative for decay)

This is also the formula for continuously compounded interest, where P_0 is the principal amount invested at an interest rate k compounded continuously, and $P(t)$ is the amount in the account after t years.

Growth Example:

Given an Initial population of 100 rabbits with a growth rate 11.7% per day,

A. The exponential growth function is $P(t) = 100e^{0.117t}$

B. The population after 7 days is $P(7) = 100e^{0.117(7)} \approx 226.8 \approx 227\ rabbits$

The population after 2 weeks is $P(14) = 100e^{0.117(14)} \approx 514.5 \approx 515\ rabbits$

C. The doubling time can be found by setting $P(t) = 2P_0$

$2P_0 = P_0 e^{0.117t}$

Dividing both sides by P_0 yields $2 = e^{0.117t}$

Taking the natural log of both sides yields $\ln 2 = 0.117t$

Dividing by 0.117 yields a doubling time of $t = \frac{\ln 2}{0.117} \approx 5.9\ days$

When an exponential decay problem refers to the half life of a substance, it means the time t for which $P(t) = \frac{1}{2}P_0$.

Solving Systems of Equations with Matrices

See http://www.asms.net/brewer/precal-matrixoperations.pdf

Arithmetic Sequences and Series

A sequence is arithmetic if there exists a number d, called the common difference, such that $a_{n+1} = a_n + d$, for any integer $n \geq 1$.

The nth term of an arithmetic sequence is given by $a_n = a_1 + (n-1)d$, for any integer $n \geq 1$.

The sum of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}(a_1 + a_n)$.

Geometric Sequences and Series

A sequence is geometric is there is a number r, called the common ratio, such that $\frac{a_{n+1}}{a_n} = r$, or $a_{n+1} = a_n r$, for any integer $n \geq 1$.

The nth term of a geometric sequence is given by $a_n = a_1 r^{n-1}$, for any integer $n \geq 1$.

The sum of the first n terms of a geometric series is given by $S_n = \frac{a_1(1-r^n)}{1-r}$, for any $r \neq 1$.

When $|r| < 1$, the limit or sum of an infinite geometric series is given by $S_\infty = \frac{a_1}{1-r}$.

Binomials and the Binomial Theorem

$n! = n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1$

"n choose k" = $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = the number of ways to choose k objects out of a set containing n objects

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$The\ (k+1)^{st}\ term\ of\ (a+b)^n\ is\ \binom{n}{k} a^{n-k} b^k$$

Patterns to be found in $(a+b)^n$:

- $(a+b)^n$ has n+1 terms
- powers of a decrease from n to 0
- powers of b increase from 0 to n
- sum of exponents in each term is n
- coefficients come from Pascal's triangle

1 = Degree 0
1 1 = Degree 1
1 2 1 = Degree 2
1 3 3 1 = Degree 3
1 4 6 4 1 = Degree 4
1 5 10 10 5 1 = Degree 5
1 6 15 20 15 6 1 = Degree 6
1 7 21 35 35 21 7 1 = Degree 7
1 8 28 56 70 56 28 8 1 = Degree 8
1 9 36 84 126 126 84 36 9 1 = Degree 9
1 10 45 120 210 252 210 120 45 10 1 = Degree 10

The total number of subsets of a set containing n elements is 2^n.