How to Solve a System of Equations in Three Variables Using an Augmented Matrix

Given the system of equations:

$$\begin{cases} 3x + 2y + 2z = 3 \\ \;\; x + 2y - \;\; z = 5 \\ 2x - 4y + \;\; z = 0 \end{cases}$$

The corresponding augmented matrix (called augmented because it contains the constant terms as well as the coefficients) is

$$\left[\begin{array}{ccc|c} 3 & 2 & 2 & 3 \\ 1 & 2 & -1 & 5 \\ 2 & -4 & 1 & 0 \end{array}\right]$$

Using Gauss-Jordan Elimination, we can produce an equivalent matrix, which is said to be in *reduced row-echelon form*, from which we can easily read off the solution to the system of equations.

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$, where the solution to the system of equations is the ordered triple (x, y, z).

To get from $\left[\begin{array}{ccc|c} 3 & 2 & 2 & 3 \\ 1 & 2 & -1 & 5 \\ 2 & -4 & 1 & 0 \end{array}\right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$, we are allowed to perform the following operations:

1. Interchange any two rows
2. Multiply each entry in a row by the same nonzero constant
3. Add a nonzero multiple of one row to another row

A strategy that will work every time is to work with the left column first, getting the 1 then the zeros; then the middle column, getting the 1 first and then the zeros; and finally the third column, getting the 1 first and then the zeros. It is efficient to get the 1's before the 0's because the 1's first make getting the 0's easier. It is also best to work from one side to the other so that correct entries stay correct when working with other entries.

Let's put the matrix $\left[\begin{array}{ccc|c} 3 & 2 & 2 & 3 \\ 1 & 2 & -1 & 5 \\ 2 & -4 & 1 & 0 \end{array}\right]$ in reduced row-echelon form.

First, we want a 1 in the top left entry. This can be accomplished in several ways. The first row can be divided by 3, the third row can be subtracted from the first row, or the first and second rows can be interchanged. The simplest action leaves less room for error, so we will interchange rows 2 and 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 2 & 2 & 3 \\ 2 & -4 & 1 & 0 \end{array}\right]$$

Next, we want to get 0's in the other two entries in the first column. This can be achieved by adding a constant multiple of the first row to each of the 2nd and 3rd rows. We will add $(-3) \cdot row\ 1$ to $row\ 2$ and $(-2) \cdot row\ 1$ to $row\ 3$.

The entries in the 2nd row change as follows:
$3 - 3 \cdot 1 = 0$
$2 - 3 \cdot 2 = -4$
$2 - 3 \cdot (-1) = 5$
$3 - 3 \cdot 5 = -12$

The entries in the 3rd row change as follows:
$2 - 2 \cdot 1 = 0$
$-4 - 2 \cdot 2 = -8$
$1 - 2 \cdot (-1) = 3$
$0 - 2 \cdot 5 = -10$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 2 & 2 & 3 \\ 2 & -4 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - 2 \cdot R_1]{R_2 - 3 \cdot R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -4 & 5 & -12 \\ 0 & -8 & 3 & -10 \end{array}\right]$$

Next, we want to get a 1 where the -4 is. The simplest way to do this is to multiply the 2nd row by the multiplicative inverse of -4, which is the same as dividing by -4.

The entries in the 2nd row change as follows:
$0 \div (-4) = 0$
$-4 \div (-4) = 1$
$5 \div (-4) = -\frac{5}{4}$
$-12 \div (-4) = 3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -4 & 5 & -12 \\ 0 & -8 & 3 & -10 \end{array}\right] \xrightarrow{R_2 \div (-4)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & -8 & 3 & -10 \end{array}\right]$$

To get 0's in the other two entries in the 2nd column, we perform similar operations as we did to get the 0's in the first column. We will add $(-2) \cdot row\ 2$ to $row\ 1$ and $(8) \cdot row\ 2$ to $row\ 3$.

The entries in the 1st row change as follows:
$1 - 2 \cdot 0 = 1$
$2 - 2 \cdot 1 = 0$
$-1 - 2 \cdot \left(-\frac{5}{4}\right) = \frac{3}{2}$
$5 - 2 \cdot 3 = -1$

If you are using a TI-30XS Multiview, use the $\frac{n}{d}$ key to enter these operations, including fractions, exactly as they appear so that there is little room for mistakes.

The entries in the 3rd row change as follows:

$0 + 8 \cdot 0 = 0$

$-8 + 8 \cdot 1 = 0$

$3 + 8 \cdot \left(-\frac{5}{4}\right) = -7$

$-10 + 8 \cdot 3 = 14$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & -8 & 3 & -10 \end{array}\right] \xrightarrow[R_3 + 8 \cdot R_2]{R_1 - 2 \cdot R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & -1 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & 0 & -7 & 14 \end{array}\right]$$

To get a 1 where the -7 is, we multiply by the multiplicative inverse, which is the same as dividing by -7.

The entries in row 3 change as follows:

$0 \div (-7) = 0$

$0 \div (-7) = 0$

$-7 \div (-7) = 1$

$14 \div (-7) = -2$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & -1 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & 0 & -7 & 14 \end{array}\right] \xrightarrow{R_3 \div (-7)} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & -1 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Finally, we have to get 0's in the other two entries in the third column. We will add $\left(-\frac{3}{2}\right) \cdot row\ 3$ to $row\ 1$ and $\left(\frac{5}{4}\right) \cdot row\ 3$ to $row\ 2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & -1 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_2 + \frac{5}{4} \cdot R_3]{R_1 - \frac{3}{2} \cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Thus, the solution to the system of equations is $\left(2, \frac{1}{2}, -2\right)$.

To put it all together, only showing the steps with arrows, the solution to this problem looks like:

$$\left[\begin{array}{ccc|c} 3 & 2 & 2 & 3 \\ 1 & 2 & -1 & 5 \\ 2 & -4 & 1 & 0 \end{array}\right] \xrightarrow[R_1 and\ R_2]{interchange} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 3 & 2 & 2 & 3 \\ 2 & -4 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - 2 \cdot R_1]{R_2 - 3 \cdot R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -4 & 5 & -12 \\ 0 & -8 & 3 & -10 \end{array}\right] \xrightarrow{R_2 \div (-4)}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & -8 & 3 & -10 \end{array}\right] \xrightarrow[R_3 + 8 \cdot R_2]{R_1 - 2 \cdot R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & -1 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & 0 & -7 & 14 \end{array}\right] \xrightarrow{R_3 \div (-7)} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & -1 \\ 0 & 1 & -\frac{5}{4} & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_2 + \frac{5}{4} \cdot R_3]{R_1 - \frac{3}{2} \cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -2 \end{array}\right]$$