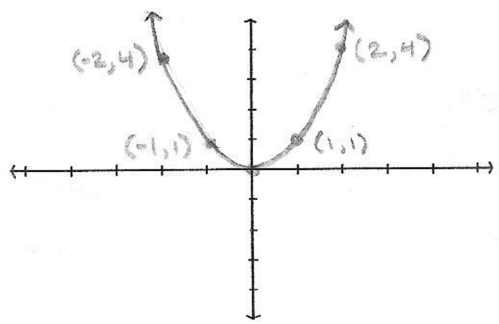
PRECALCULUS – GUIDE TO GRAPHING

Sarah Gelsinger Brewer – Alabama School of Math and Science 2009

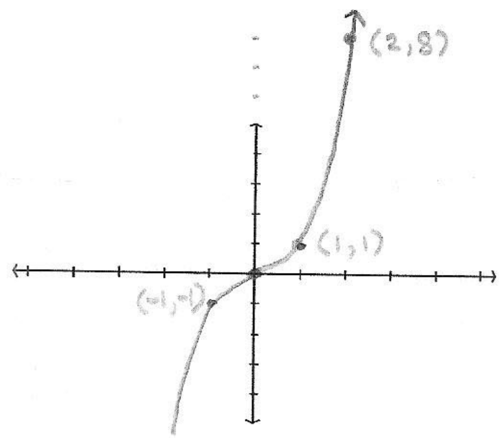
Graphing by transformation

Basic algebraic functions:

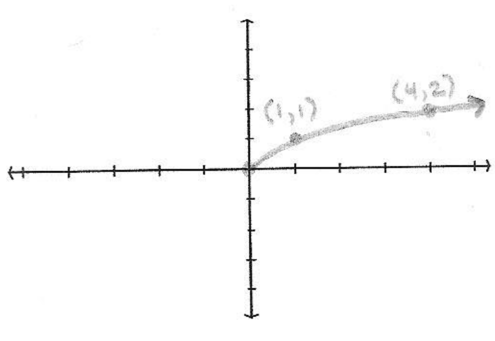
$y = x^2$



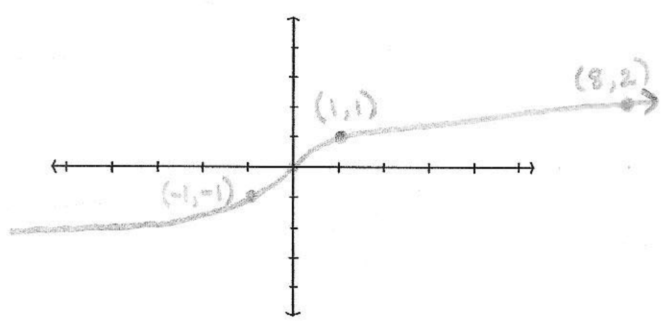
$y = x^3$



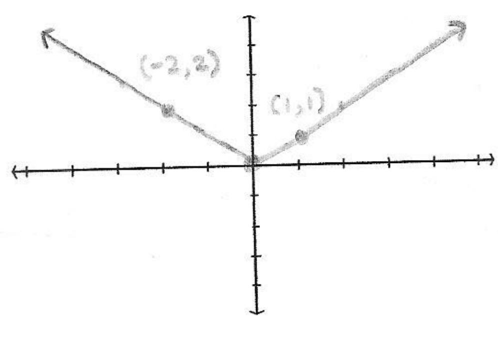
$y = \sqrt{x}$



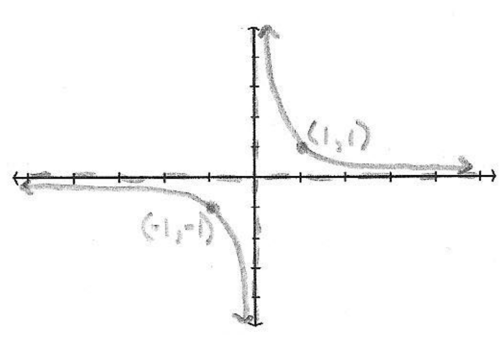
$y = \sqrt[3]{x}$



$y = |x|$



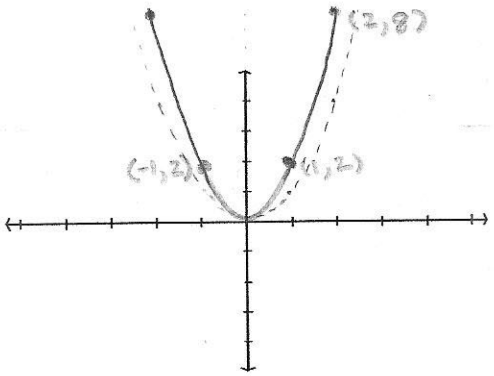
$y = \frac{1}{x}$



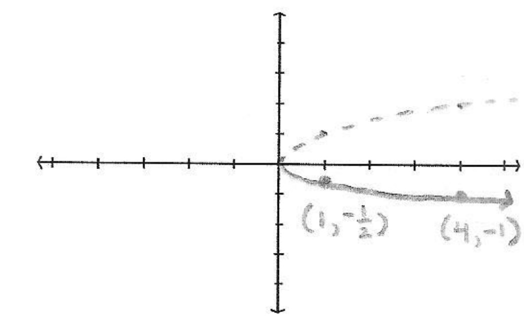
$y = af(x)$

A constant a multiplied outside a function results in a vertical shrink or stretch. Graph $y = f(x)$ first. If $|a| > 1$, stretch the graph vertically by a factor of a. If $|a| < 1$, shrink the graph vertically by a factor of a. If $a < 0$, flip the graph vertically. To apply the transformation, multiply the y-values at each of your reference points by a.

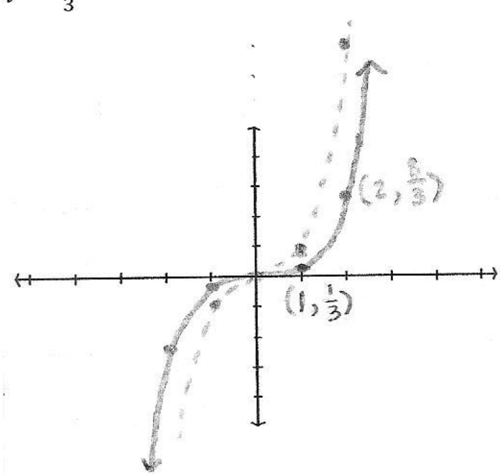
Ex. $y = 2x^2$



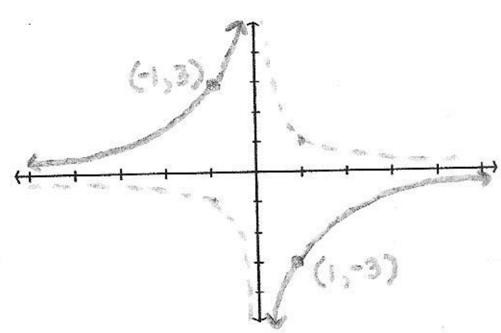
Ex. $y = -\frac{1}{2}\sqrt{x}$



Ex. $y = \frac{1}{3}x^3$



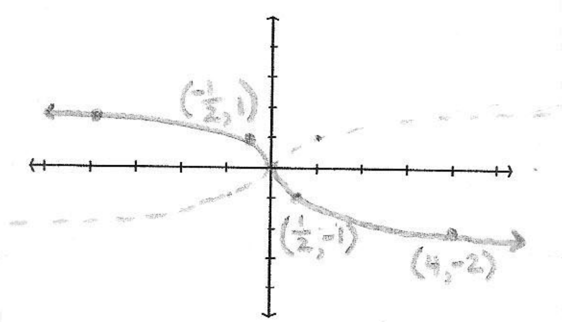
Ex. $y = -\frac{3}{x}$ $\left(= -3\frac{1}{x}\right)$



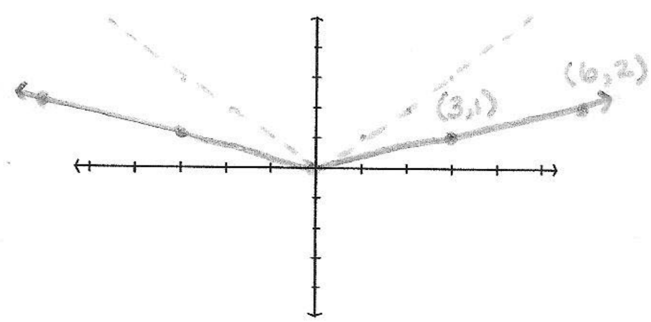
$y = f(bx)$

A constant b multiplied inside a function results in a horizontal shrink or stretch. Graph $y = f(x)$ first. If $|b| > 1$, shrink the graph horizontally by a factor of b. If $|b| < 1$, stretch the graph horizontally by a factor of b. Note that this is the opposite of what happens for vertical stretching/shrinking. If $b < 0$, flip the graph horizontally. To apply the transformation, divide the x-values at each of your reference points by b.

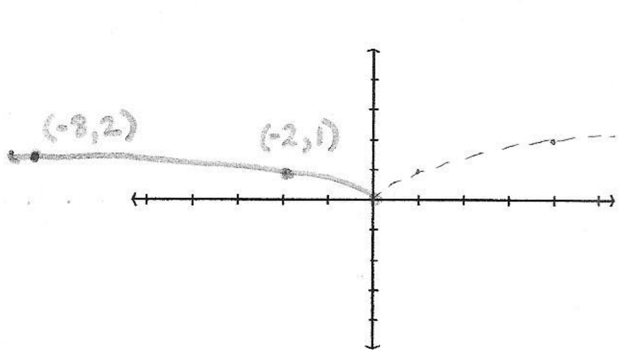
Ex. $y = \sqrt[3]{-2x}$



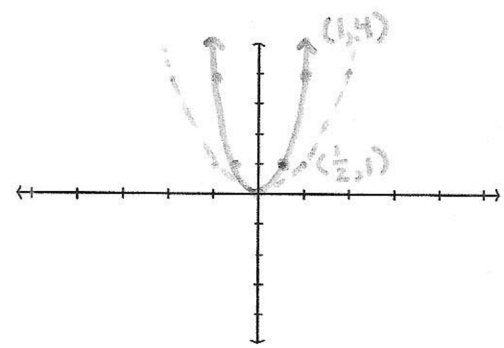
Ex. $y = \left|\frac{1}{3}x\right|$



Ex. $y = \sqrt{-\frac{1}{2}x}$ $\left(= \sqrt{-\frac{x}{2}}\right)$



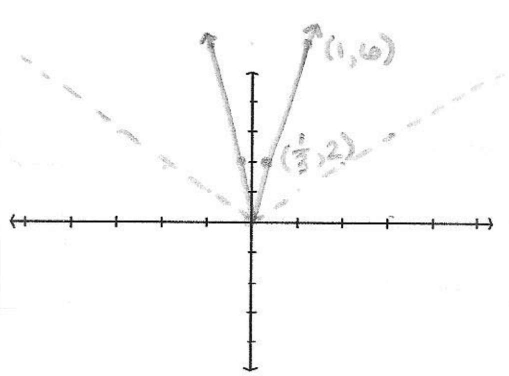
Ex. $y = (2x)^2$



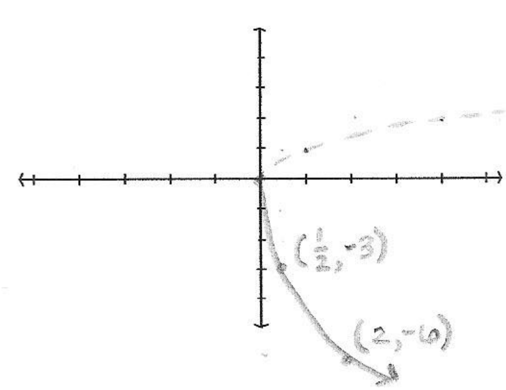
$\boldsymbol{y = af(bx)}$

Combine instructions for $y = af(x)$ and $y = f(bx)$ above. At each reference point, multiply the y-value by a and divide the x-value by b.

Ex. $y = 2|3x|$



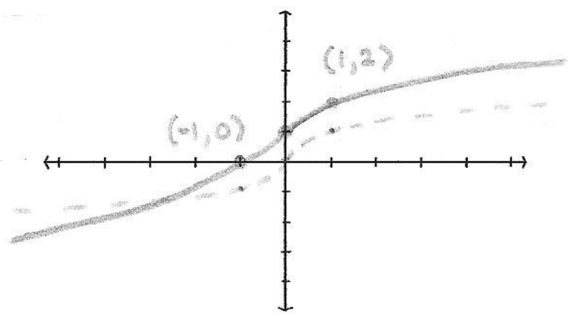
Ex. $y = -3\sqrt{2x}$



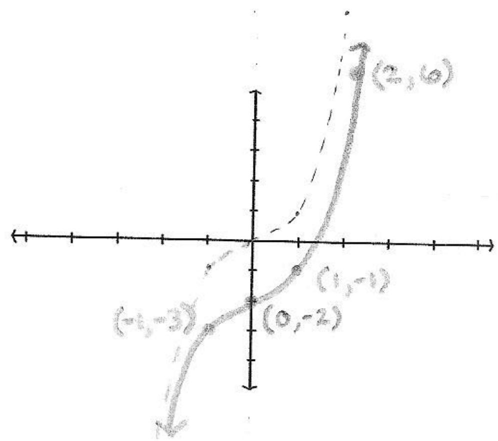
$y = f(x) + d$

A constant added outside a function results in a vertical shift. If $|d| > 1$, shift up by d units. If $|d| < 1$, shift down by d units. To apply the transformation, add d to the y-value at each reference point. Note: when shifting $y = \frac{1}{x}$, be sure to shift the asymptotes.

Ex. $y = \sqrt[3]{x} + 1$



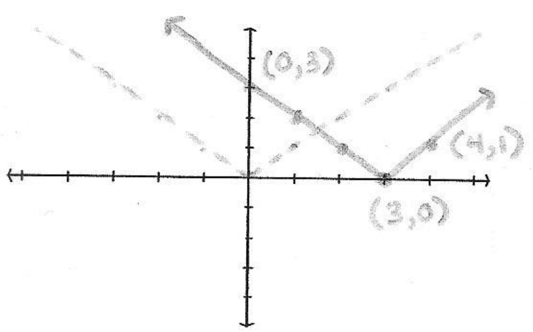
Ex. $y = x^3 - 2$



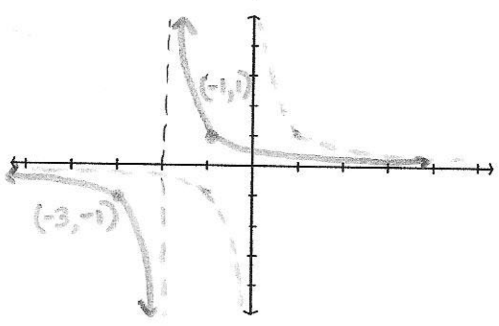
$y = f(x + c)$

A constant added inside a function results in a horizontal shift. If $|c| > 1$, shift left by c units. If $|c| < 1$, shift right by c units. To apply the transformation, subtract c from the x-value at each reference point. Note: when shifting $y = \frac{1}{x}$, be sure to shift the asymptotes.

Ex. $y = |x - 3|$



Ex. $y = \frac{1}{x+2}$



$y = f(x + c) + d$

Combine instructions for $y = f(x) + d$ and $y = f(x + c)$ above. Shift each reference point by subtracting c from the x-value and adding d to the y-value.

$$y = af(bx + c) + d = af\left[b\left(x + \frac{c}{b}\right)\right] + d$$

When combining shrinking/stretching with shifting (multiplication with addition), always perform the shrinking/stretching first. If both b and c are present, rewrite the function by factoring out b. Now, b is still a horizontal stretch, but the horizontal shift is actually $\frac{c}{b}$ and NOT just c.

****Four steps to graphing a simple algebraic function with all possible transformations of the form**

$$y = af(bx + c) + d = af\left[b\left(x + \frac{c}{b}\right)\right] + d$$

1. Graph the basic function $y = f(x)$ and label enough reference points to determine the shape of the graph (x, y)

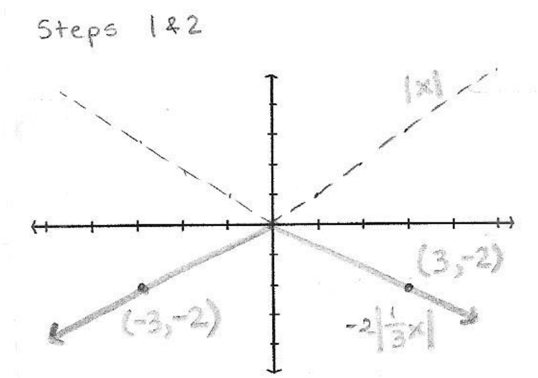
2. Multiply the y-values by a and divide the x values by b, so that now your reference points look like $\left(\frac{x}{b}, ay\right)$

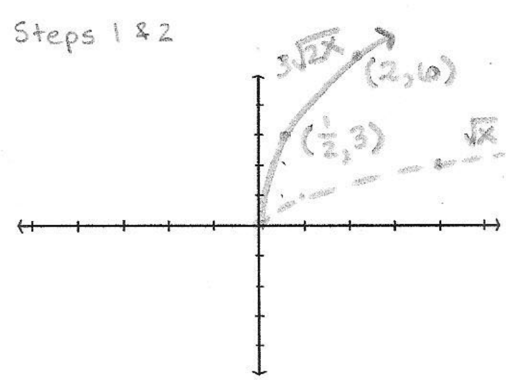
3. Add d to the y-values and subtract $\frac{c}{b}$ from the x-values so that your new reference points look like $\left(\frac{x}{b} - \frac{c}{b}, ay + d\right)$

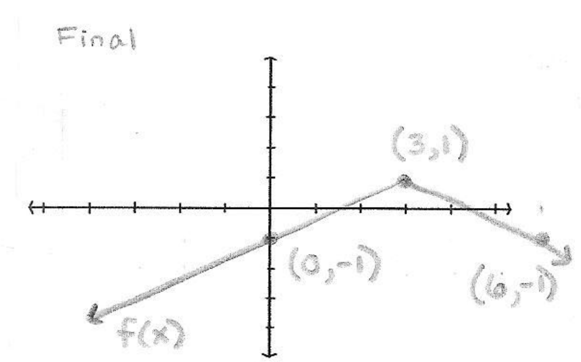
4. Use new reference points to draw the final graph $y = af(bx + c) + d$

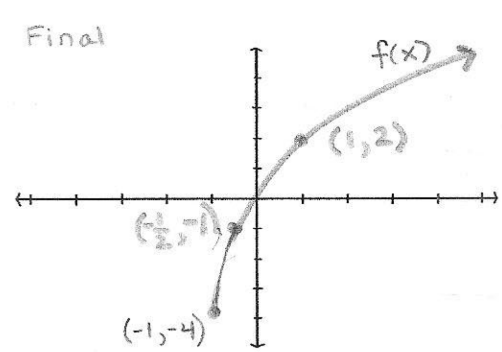
Ex. $f(x) = -2\left|\frac{1}{3}x - 1\right| + 1$ $\left(= -2\left|\frac{1}{3}(x - 3)\right| + 1\right)$

Ex. $f(x) = 3\sqrt{2x + 2} - 4$ $\left(= 3\sqrt{2(x + 1)} - 4\right)$









Piecewise Functions

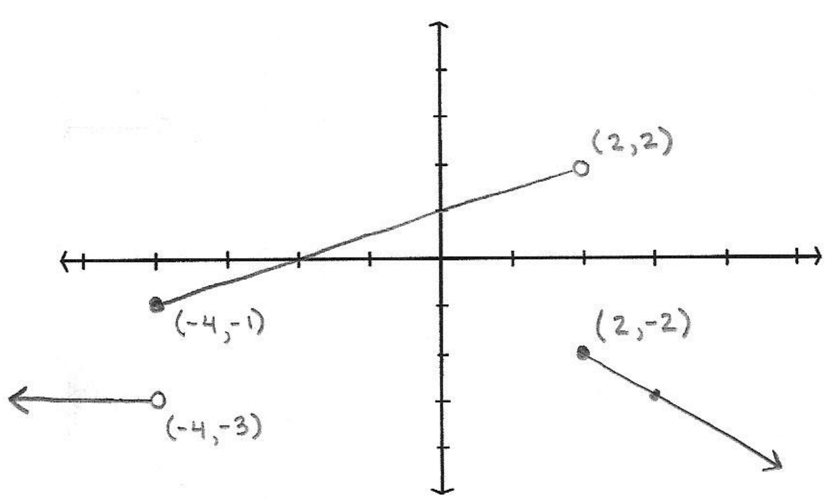
Graphing a piecewise function is essentially like graphing several different functions, selecting different pieces of each one, and then placing them all on the same graph, making sure that the intervals of x-values don't overlap so that the end result is still a function. The key is to look at the endpoints of the x-value intervals. Plug those values into the corresponding function and plot the points. If a strict inequality is used ($<$ *or* $>$), draw an open circle. If $\leq$ *or* $\geq$ is used, draw a filled-in circle.

Ex. $f(x) = \begin{cases} -3 & for\ x < -4 \\ \frac{1}{2}x + 1 & for -4 \leq x < 2 \\ -x & for\ x \geq 2 \end{cases}$

First, we will look at $f(x) = -3$, whose domain has been restricted to x-values strictly less than (to the left of) -4. This is a constant function, so no matter what x-value is plugged in, the output will be the y-value -3. Here we have a strict inequality, so we will plot the point $(-4, -3)$ with an open circle. We want to continue this function for all x-values to the left of -4, so draw a horizontal line from the point $(-4, -3)$ to the left-hand edge of the graph.

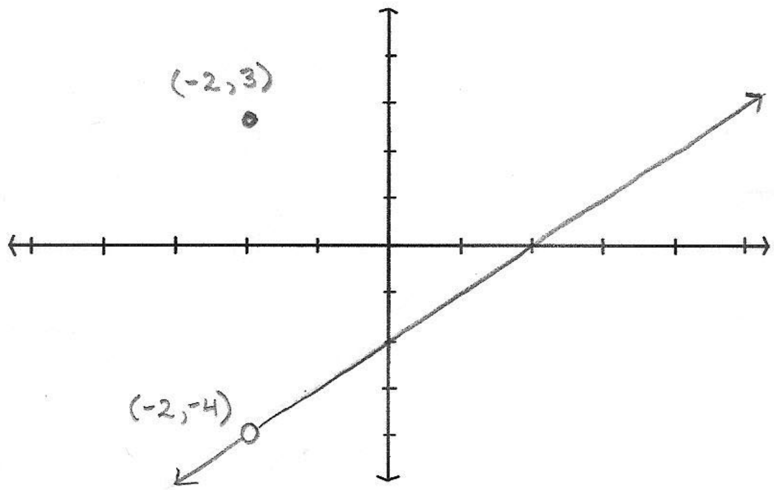
Next, Let's look at $f(x) = \frac{1}{2}x + 1$ whose domain has been restricted to x-values between -4 and 2, including -4 but not including 2. Note that this is the equation of a straight line with y-intercept 1 and slope $\frac{1}{2}$. We need two points in order to graph a straight line. Here, we find the endpoints of the interval by plugging -4 and 2 into the function. The output of -4 is -1 and the output of 2 is 2, so we plot the point $(-4, -1)$ with a filled-in circle and the point $(2, 2)$ with an open circle. Draw a straight line connecting the points.

Lastly, let's look at $f(x) = -x$. This is a straight line with slope -1 and y-intercept 0. Plot the endpoint of the interval by plugging 2 in for x, resulting in the point $(2, -2)$, which we draw with a filled-in circle. We still need one more point to define the line, so plug in an x-value greater than 2, say 3, to get the point $(3, -3)$. Draw a straight line from the point $(2, -2)$ through the point $(3, -3)$ extending to the edge of the graph.



Ex. $f(x) = \begin{cases} \frac{x^2-4}{x+2}, & for\ x \neq -2 \\ 3, & for\ x = -2 \end{cases}$

Note that if we factor the first function, we get $f(x) = \frac{(x-2)(x+2)}{x+2}$, and as long as $x \neq -2$, we can cancel the $x + 2$'s. Thus, this function is equal to the function $f(x) = x - 2$ everywhere except for $x = -2$. Its graph will look like the line $y = x - 2$, except it will have an open hole at the point $(-2, -4)$. The second part of the piecewise function gives us just a single filled-in point at $(-2, 3)$.



Polynomials

Lead Term Test

This determines the end behavior of the graph. Look at the degree of the leading term, and determine whether the leading coefficient is positive or negative. If the degree is even and the leading coefficient is positive, the function will behave like $y = x^2$ as x approaches $\pm\infty$. If the degree is even and the leading coefficient is negative, the function will behave like $y = -x^2$ as x approaches $\pm\infty$. If the degree is odd and the leading coefficient is positive, the function will behave like $y = x^3$ as x approaches $\pm\infty$. If the degree is odd and the leading coefficient is negative, the function will behave like $y = -x^3$ as x approaches $\pm\infty$.

Finding Zeros

Set the polynomial equal to zero and solve for x. The zeros are the x-coordinates of the x-intercepts. If the polynomial can be factored, each factor can be set equal to zero. The power of each factor is called the **multiplicity**. If a zero has odd multiplicity, the graph passes through the x-axis at that zero. If a zero has even multiplicity, the graph will touch and bounce off the graph at that zero.

Finding the y- intercepts

A function crosses the y-axis when x=0, so plug 0 in for x in the expression. $(0, f(0))$ is the y-intercept.

Rational Functions

Finding Zeros

Set numerator equal to zero and solve for x. The zeros are the x-coordinates of the x-intercepts.

Finding Vertical Asymptotes

Set denominator equal to zero and solve for x. Any factors that cancel with factors in the numerator will result in holes in the graph rather than asymptotes.

Finding Horizontal and Oblique Asymptotes

These determine the end behavior of the graph. Look at the ratio of lead terms in the numerator and denominator. If the degree of the denominator is higher than the degree in the numerator, there will be a horizontal asymptote at $y = 0$. If the degrees are the same, then there will be a horizontal asymptote at $= \frac{p}{q}$, where $\frac{p}{q}$ is the ratio of leading coefficients. If the degree of the numerator is one higher than the degree of the denominator, then there will be an oblique (linear) asymptote. To find it, perform long division and stop after you find the constant term (ignore the remainder).

Finding y- intercepts

A function crosses the y-axis when x=0, so plug 0 in for x in the expression. $(0, f(0))$ is the y-intercept.