Topology HW #4: Ch 2 Section 13, p. 83 #1,3,7,8

**1.** Let $X$ be a topological space; let $A$ be a subset of $X$. Suppose that for each $x \in A$ there is an open set $U$ containing $x$ such that $U \subset A$. Show that $A$ is open in $X$.

Hint: use basis criteria

**3.** Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set $X$. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

Hint: $\mathcal{T}_c$ is the co-countable topology described below. Use definition of a topology and theorems from Ch 7 about countability to prove. $\mathcal{T}_\infty$ is not a topology, but you should still work through the definition to try to prove it until you run into something that you can't prove, in which case you need to construct a SPECIFIC counter-example to show that the particular criteria fails for this collection.

**7.** Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1 =$ the standard topology,
$\mathcal{T}_2 =$ the topology of $\mathbb{R}_K$,
$\mathcal{T}_3 =$ the finite complement topology,
$\mathcal{T}_4 =$ the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5 =$ the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

Hint: use definitions of comparable, finer, and coarser along with our standard method of proving that one set is a subset of another to show inclusions. If two sets are not comparable, construct specific counter-examples.

**8.** (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Hint: see Lemma 13.2 below for (a). For (b), see criteria for a basis. To show that this topology is different from the lower limit topology, try to compare the two.

A set A is **finite** if it is in bijection with a finite subset of $\mathbb{Z}_+$, i.e. there exists a bijection $f: A \to \{1, 2, 3, \dots, n\}$ for some $n \in \mathbb{Z}_+$.

**Cor 6.7** Let $B \neq \emptyset$. The following are equivalent:

1) $B$ is finite
2) there exists a surjection $f: \{1, 2, 3, \dots, n\} \to B$
3) there exists an injection $f: B \to \{1, 2, 3, \dots, n\}$

A set is **infinite** if it is not finite. A set $A$ is said to be **countably infinite** if there exists a bijection $f: A \to \mathbb{Z}_+$.

$A$ is said to be **countable** if it is finite or countably infinite. Else it is said to be **uncountable**.
Note: $\emptyset$ is finite and therefore countable

**Thm 7.1** Let $B \neq \emptyset$. The following are equivalent:

1) $B$ is countable
2) there exists a surjection $f: \mathbb{Z}_+ \to B$
3) there exists an injection $f: B \to \mathbb{Z}_+$

**Lemma 7.2** If C is any infinite subset of $\mathbb{Z}_+$, then C is countably infinite.

**Cor 7.3** Every subset of a countable set is countable.

**Thm 7.5** Countable union of countable sets is countable

A **topology** on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ with the following properties:

1) $\emptyset, X$ are in $\mathcal{T}$
2) the union of elements in *any* (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$
$$\bigcup_\alpha U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$
3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$
$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

A set $X$ with a specified $\mathcal{T}$ is called a **topological space**, denoted by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. $U$ is said to be **open** if $U \in \mathcal{T}$.

$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.
If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, and that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$.

If $X$ is a set, a **basis** for a topology on $X$ is a collection $\mathfrak{B}$ of subsets of $X$ such that:

1) for every $x \in \mathrm{X}$, $\exists\, \mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B}$
2) Given $\mathrm{B}_1, \mathrm{B}_2 \in \mathfrak{B}$, if $x \in \mathrm{B}_1 \cap \mathrm{B}_2$, then $\exists\, \mathrm{B}_3 \in \mathfrak{B}$ such that $x \in \mathrm{B}_3 \subset \mathrm{B}_1 \cap \mathrm{B}_2$

A subset $U$ of $X$ is said to be **open** in $X$ (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $\mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B} \subset U$.

$\mathfrak{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathfrak{B}$, i.e. for any $U \subseteq X$, $U$ open, and $x \in U$, $\exists\, \mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B} \subset U$.

**discrete topology**: the collection of all subsets of $X$ (same as power set)

**indiscrete or trivial topology**: the collection consisting of $X$ and $\emptyset$ only

**finite complement topology**: the collection $\mathcal{T}_f$ of all subsets $U$ of $X$ such that $X - U$ is either finite or all of $X$

**co-countable topology**: the collection $\mathcal{T}_C$ of all subsets $U$ of $X$ such that $X - U$ is either countable or all of $X$

Let $X$ be a topological space and $Y \subset X$. Then the collection $\mathcal{T}_Y = \{U \cap X | U \text{ open in } X\}$ is a topology on Y, called the **subspace topology**.

**standard topology** on $\mathbb{R}$: topology generated by all open intervals $(a, b) = \{x \in \mathbb{R} | a < x < b\}$; whenever we consider $\mathbb{R}$, we assume it is given this topology unless we specifically state otherwise

**the lower-limit topology** on $\mathbb{R}$: topology generated by the collection $\mathbb{R}_\ell$ all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$

**K-topology** on $\mathbb{R}$: the topology generated by the collection $\mathbb{R}_K$ of all open intervals $(a, b)$ along with all sets of the form $(a, b) - K$, where $K$ is the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$

**Lemma 13.1**: Let $X$ be a set; let $\mathfrak{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

**Lemma 13.2**: Let $X$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of $X$ such that for each open set $U$ of $X$ and each $x$ in $U$, there is an element $C$ of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of $X$.

**Lemma 13.3**: Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on $X$. Then the following are equivalent:

1) $\mathcal{T}'$ is finer than $\mathcal{T}$
2) For each $x \in X$, and each basis element $B \in \mathfrak{B}$ containing $x$, there is a basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subseteq B$