Closed Sets, Limit Points, and Continuity

Def Let X be a topological space. $A \subset X$ is a **closed set** if $X - A$ ia open.

Thm 17.1 Let X be a topological space. The following hold:

1) $\emptyset$, X are closed
2) arbitrary intersections of closed sets are closed, i.e. if A_i are closed, $\bigcap_i A_i$ is closed.
3) finite unions of closed sets are closed

Thm 17.2 Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Def Let X be a topological space and let $A \subset X$. The **closure** of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

Thm 17.5 Let $A \subset X$and let $\mathfrak{B}$ be a basis for X. Then $x \in \bar{A}$ if and only if every open set U containing x intersects A and $x \in \bar{A}$ if and only if every basis element B containing x intersects A

Def U is a **neighborhood** of x if U is open and $x \in U$

Def Let X be a topological space and let $A \subset X$. $x \in X$ is said to be a **limit point** of A if every neighborhood of x intersects A in a point other than x.

Thm 17.6 Let $A \subset X$ and let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Cor 17.7 A is closed if and only if A contains all its limit points.

Def A topological space is **Hausdorff** if for each pair of distinct points x and y, there exist disjoint neighborhoods of x and y.

Thm 17.8 Every finite set in a Hausdorff space is closed.

Thm 17.9 Let X be a Hausdorff space and let $A \subset X$. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Def Let X and Y be topological spaces and $f: X \to Y$ be a function. f is said to be **continuous** if for every open set V in Y, $f^{-1}(V)$ is open in X.
$f^{-1}(V) = \{x \in X | f(x) \in V\}$

Thm 18.1 Let X and Y be topological spaces and $f: X \to Y$ be a function. The following are equivalent:

1) f is continuous
2) $f^{-1}(B)$ is closed for every closed set $B \subset Y$
3) for each $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.

Def Let $f: X \to Y$ be a bijection. If both f and f^{-1} are continuous, then f is called a **homeomorphism**.