Topology Definitions & Selected Notes

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Section numbers refer to Munkres, *Topology*, 2nd ed.

1.1 - Fundamental Concepts, Set Theory, & Logic

We will use capital letters $A, B, \dots$ to denote sets, and lowercase letters $a, b, \dots$ to denote the objects or elements belonging to these sets.
If an object a belongs to a set A, we express this by the notation $a \in A$.
If an object a does not belong to A, we express this by the notation $a \notin A$.

We say that A is a subset of B if every element of A is also an element of B, and express this by writing $A \subseteq B$. Note that A is not required to be different from B. If $A = B$, then we have both $A \subseteq B$ and $B \subseteq A$. If $A \subseteq B$ and $A \neq B$, then we call A a proper subset of B and write $A \subset B$.
The relations $\subseteq$ and $\subset$ are called inclusion and proper inclusion. If $A \subseteq B$, we can also express this by $B \supseteq A$.

A set with a small finite number of elements can be expressed by the roster method, e.g. $A = \{a, b, c\}$

Typically, we will specify a set by one element of the set and some property that elements of the set may or may not possess, to form the set of all elements of the set having that property. e.g. $B = \{x | x \text{ is an even integer}\}$ This method is often called set-builder notation.

The set having no elements is called the empty set, denoted by $\emptyset$.

If sets A and B have no common elements, we say that they are disjoint, and express this by $A \cap B = \emptyset$.

For every set A, we have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

The **union** of two sets A and B is $A \cup B = \{x | x \in A \text{ or } x \in B\}$.
The **intersection** of two sets A and B is $A \cap B = \{x | x \in A \text{ and } x \in B\}$.
The **difference** of two sets A and B is $A - B = \{x | x \in A \text{ and } x \notin B\}$.
An **arbitrary union** is $\bigcup_{A \in \mathcal{A}} A = \{x | x \in A \text{ for at least one } A \in \mathcal{A}\}$.
An **arbitrary intersection** is $\bigcap_{A \in \mathcal{A}} A = \{x | x \in A \text{ for every } A \in \mathcal{A}\}$.
The **power set** of A, denoted by $\mathcal{P}(A)$ is the set of all subsets of A.
The **cartesian product** of sets A and B is $A \times B = \{(x, y) | x \in A \text{ and } y \in B\}$.
For a statement "If A, then B," the **converse** is "If B, then A" and the **contrapositive** is "If not B, then not A."
A statement "There exists an element $a \in A$ such that P is true" has **negation** "For all elements $a \in A$, P is not true."
A statement "For all elements $a \in A$, P is true" has **negation** "There exists an element $a \in A$ such that P is not true."
$x \notin A \cup B \Rightarrow x \notin A \text{ and } x \notin B$; $x \notin A \cap B \Rightarrow x \notin A \text{ or } x \notin B$
$(x, y) \notin A \times B \Rightarrow x \notin A \text{ or } y \notin B$; $x \notin A - B \Rightarrow x \notin A \text{ or } x \in B$
To show that a statement is true, prove it in general; to show that a statement is false, provide a specific counter-example.
To prove that two sets are equal ($A = B$), prove both $A \subseteq B$ and $B \subseteq A$.
To show $A \subseteq B$, choose an arbitrary element $x \in A$ and build an argument that leads to $x \in B$.

Distributive Law for Unions & Intersections:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

DeMorgan's Laws:

$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$

1.2 – Functions

A **function** $f: A \to B$ is a rule of assignment (which assigns to every input exactly one output) together with a set B (range) that contains the image set of the rule.

$f: A \to B$ is **injective** (or one-to-one) if given any elements $a_1, a_2 \in A, f(a_1) = f(a_2)$ implies that $a_1 = a_2$. This definition can also be stated as its contrapositive: Given any elements $a_1, a_2 \in A$ such that $a_1 \neq a_2$, if this implies that $f(a_1) \neq f(a_2)$, then f is injective.

f is **surjective** (or onto) if given any $b \in B$, there exists some $a \in A$ such that $f(a) = b$. (i.e., if given any element in the set B, you can find some element in the set A that maps to it.)

f is **bijective** if it is both injective and surjective. If f is a bijective function, then there exists a function $f^{-1}: B \to A$ called the **inverse function** of f, defined as $f^{-1}(b) = a$ if and only if $f(a) = b$.

Given a function $f: A \to B$, and $A_0 \subseteq A$, the **image** of A_0 under f is $f(A_0) = \{b \in B | b = f(a) for\ at\ least\ one\ a \in A_0\}$
.

Given a function $f: A \to B$, and $B_0 \subseteq B$, the **preimage** of B_0 under f is $f^{-1}(B_0) = \{a \in A | f(a) \in B_0\}$. Note that if there are no points a of A whose images lie in B_0, then the set $f^{-1}(B_0)$ is empty.

Given functions $f: A \to B$ and $g: B \to C$, the **composition** of f and g, denoted by $g \circ f$, is defined as $(g \circ f)(a) = g(f(a))$. Formally, $g \circ f: A \to C$ is the function whose rule is $\{(a, c) | For\ some\ b \in B, f(a) = b\ and\ g(b) = c\}$.

Lemma 2.1: Let $f: A \to B$ be a function. If $\exists\ g: B \to A$ & $h: B \to A$ such that $g(f(a)) = a \ \ \forall a \in A$ and $f(h(b)) = b \ \ \forall b \in B$, then f is bijective and $g = h = f^{-1}$
Translation: If a function has an inverse, then that function is bijective.

Example Problem #1(b):
Let $f: A \to B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$. Show that $f(f^{-1}(B_0)) \subseteq B_0$ and that equality holds if f is surjective (onto).

Proof:

$\subseteq$ **Let $\boldsymbol{y \in f\left(f^{-1}(B_0)\right)}$.**
< Let y be an element in the image of the set $f^{-1}(B_0)$ under f. >
Then $\boldsymbol{y = f(a)}$ for some $\boldsymbol{a \in f^{-1}(B_0)}$.
< There exists an element a in the preimage of the set B_0 under f such that $y = f(a)$. >
$\boldsymbol{\Longrightarrow f(a) \in B_0}$.
< By definition, since a is an element of the preimage of B_0 under f, this implies that the image of a under f lies in B_0. >
Since $\boldsymbol{y = f(a)}$, we have $\boldsymbol{y \in B_0}$ and hence $\boldsymbol{f\left(f^{-1}(B_0)\right) \subseteq B_0}$.

Assuming, in addition, that $\boldsymbol{f}$ is surjective:

$\supseteq$ **Let $\boldsymbol{x \in B_0}$.**
Since $\boldsymbol{B_0 \subseteq B}$ and $\boldsymbol{f}$ is surjective from $\boldsymbol{A}$ onto $\boldsymbol{B}$, there exists at least one $\boldsymbol{a \in A}$ such that $\boldsymbol{f(a) = x}$.
$\boldsymbol{f(a) = x \implies f(a) \in B_0 \implies a \in f^{-1}(B_0)}$
< Since the image of the element a under f is an element of the set B_0, this is precisely what it means for an element to be
in the preimage of the set B_0 under f. >
By definition of the image of set $\boldsymbol{f^{-1}(B_0)}$ under $\boldsymbol{f}$, we have that $\boldsymbol{f(a) \in f\left(f^{-1}(B_0)\right)}$.
Since $\boldsymbol{x = f(a)}$, we have $\boldsymbol{x \in f\left(f^{-1}(B_0)\right)}$ and hence $\boldsymbol{B_0 \subseteq f\left(f^{-1}(B_0)\right)}$.

2.12-13 – Topological Spaces

A set A is **finite** if it is in bijection with a finite subset of $\mathbb{Z}_+$, i.e. there exists a bijection $f: A \to \{1,2,3,\dots,n\}$ for some $n \in \mathbb{Z}_+$.

Cor 6.7 Let $B \neq \emptyset$. The following are equivalent:

1) B is finite
2) there exists a surjection $f: \{1,2,3,\dots,n\} \to B$
3) there exists an injection $f: B \to \{1,2,3,\dots,n\}$

A set is **infinite** if it is not finite. A set A is said to be **countably infinite** if there exists a bijection $f: A \to \mathbb{Z}_+$.

A is said to be **countable** if it is finite or countably infinite. Else it is said to be **uncountable**.
Note: $\emptyset$ is finite and therefore countable

Thm 7.1 Let $B \neq \emptyset$. The following are equivalent:

1) B is countable
2) there exists a surjection $f: \mathbb{Z}_+ \to B$
3) there exists an injection $f: B \to \mathbb{Z}_+$

Lemma 7.2 If C is any infinite subset of $\mathbb{Z}_+$, then C is countably infinite.

Cor 7.3 Every subset of a countable set is countable.

Thm 7.5 Countable union of countable sets is countable

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:

1) $\emptyset$, X are in $\mathcal{T}$
2) the union of elements in *any* (arbitrary) subcollection of $\mathcal{T}$ is in $\mathcal{T}$
$$\bigcup_{\alpha} U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$
3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$
$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

A set X with a specified $\mathcal{T}$ is called a **topological space**, denoted by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. U is said to be **open** if $U \in \mathcal{T}$.

$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.
If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$, and that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$.

If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X such that:

1) for every $x \in \mathrm{X}$, $\exists\ \mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B}$
2) Given $\mathrm{B}_1, \mathrm{B}_2 \in \mathfrak{B}$, if $x \in \mathrm{B}_1 \cap \mathrm{B}_2$, then $\exists\ \mathrm{B}_3 \in \mathfrak{B}$ such that $x \in \mathrm{B}_3 \subset \mathrm{B}_1 \cap \mathrm{B}_2$

A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $\mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B} \subset U$.

$\mathfrak{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathfrak{B}$, i.e. for any $U \subseteq X$, U open, and $x \in U$, $\exists\ \mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B} \subset U$.

discrete topology: the collection of all subsets of X (same as power set)

indiscrete or trivial topology: the collection consisting of X and $\emptyset$ only

finite complement topology: the collection $\mathcal{T}_f$ of all subsets U of X such that $X - U$ is either finite or all of X

co-countable topology: the collection $\mathcal{T}_C$ of all subsets U of X such that $X - U$ is either countable or all of X

Let X be a topological space and $Y \subset X$. Then the collection $\mathcal{T}_Y = \{U \cap X | U \text{ open in } X\}$ is a topology on Y, called the **subspace topology**.

standard topology on $\mathbb{R}$: topology generated by all open intervals $(a, b) = \{x \in \mathbb{R} | a < x < b\}$; whenever we consider $\mathbb{R}$, we assume it is given this topology unless we specifically state otherwise

the lower-limit topology on $\mathbb{R}$: topology generated by the collection $\mathbb{R}_\ell$ all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} | a \le x < b\}$

K-topology on $\mathbb{R}$: the topology generated by the collection $\mathbb{R}_K$ of all open intervals (a, b) along with all sets of the form $(a, b) - K$, where K is the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$

Lemma 13.1: Let X be a set; let $\mathfrak{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

Lemma 13.2: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 13.3: Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on X. Then the following are equivalent:

1) $\mathcal{T}'$ is finer than $\mathcal{T}$
2) For each $x \in X$, and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$ such that x $\in B' \subseteq B$

Classification of Surfaces

Notation:
$\chi(S)$ - Euler characteristic of the surface
b - # of boundary components (holes, perforations)
g - genus of the surface (number of handles or crosscaps)
v - # of vertices
e - # of edges
f - # of faces

Formulas:
$\chi(S) = v - e + f$ (for any surface)
$\chi(S) = 2 - 2g - b$ (for orientable surfaces)
$\chi(S) = 2 - g - b$ (for non-orientable surfaces)

Steps to Classify a Surface:

For the given word, label the edges (with orientation) and vertices of the corresponding polygon.

Determine the number of vertices (# of different capital letters used to label vertices in polygon), edges (# of different lowercase letters in word), and faces (# of words), and use these values to calculate the Euler characteristic.
$\boldsymbol{\chi(S)} = v - e + f$

Determine if the surface is orientable or non-orientable.
Does the surface include a Mobius band? If not, it's orientable; if so, it's non-orientable.

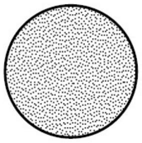
Sketch the unassociated edges with connectivity and use this to determine the number of boundary components.

Use the Euler characteristic, orientabiliy, and number of boundary components to calculate the genus of the surface. Note that the genus is always a nonnegative integer.
If orientable, $\chi(S) = 2 - 2\boldsymbol{g} - b$, or, rearranged to solve for g, $g = \frac{2-b-\chi(S)}{2}$
If non-orientable, $\chi(S) = 2 - \boldsymbol{g} - b$, or, rearranged to solve for g, $g = 2 - b - \chi(S)$

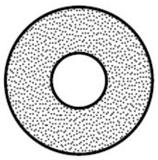
Use the genus and number of boundary components to describe the number of handles or cross-handles and holes that the surface has, and if possible, give the common name for the surface. Finally, sketch the surface.

<u>Common surfaces</u>:

Disk
orientable, no handles, genus 0, 1 boundary component



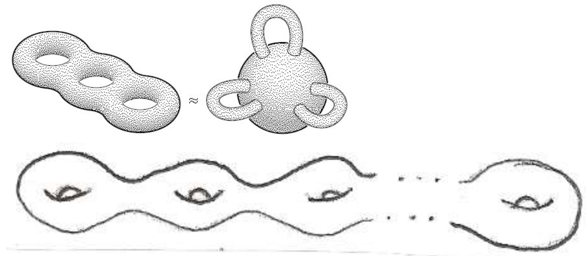
Annulus
orientable, no handles, genus 0, 2 boundary components



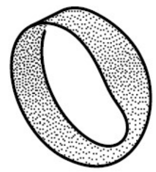
Sphere
orientable, no handles, genus 0

Torus
orientable, one handle, genus 1

2-torus, 3-torus, … K-torus
orientable, k handles, genus k

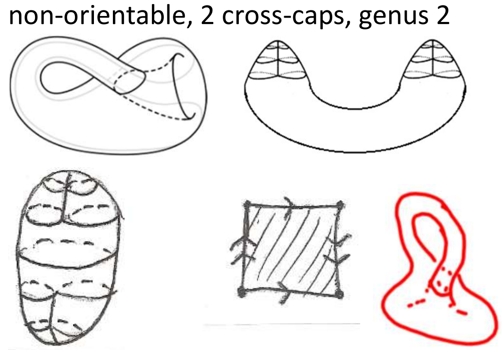


Mobius band
non-orientable, no handles, genus 1, 1 boundary component



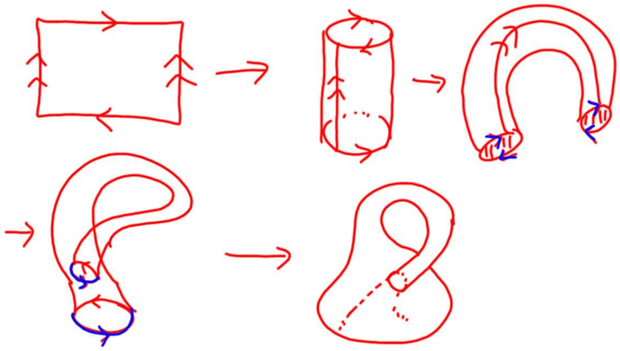
Projective plane
non-orientable, one cross-cap, genus 1

Klein bottle
non-orientable, 2 cross-caps, genus 2



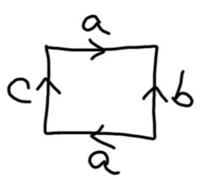
By identifying opposite sides of a rectangle, you can create an orientable surface of genus 1, the torus.

If the orientations don't match up for one of the pairs of sides, the cylinder must pass through itself, resulting in a nonorientable surface of genus 2, the Klein bottle.



All we needed to create/identify these two surfaces was a knowledge of which edges were identified with each other and with what orientation.

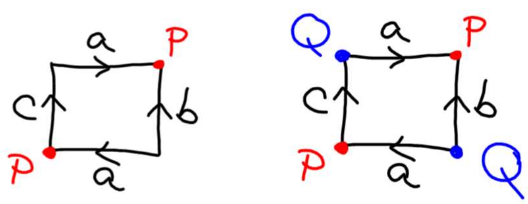
Side identification and orientation can be given in the form of a word, e.g. $ab^{-1}ac$. Four letters tell us to draw a square, one letter for each side. Start at one vertex, and write the word around the polygon. The -1 exponent on the b indicates opposite orientation. It does not matter whether you write the word clockwise or counter-clockwise, but orientations must be consistent.



Sides with the same letter are identified.

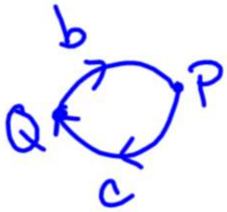
Now, we label the vertices. Choose one vertex and call it P. Since this point is at the head of edge c and the tail of edge a, any other vertex at the head of an edge c (or c^{-1}) or the tail of an edge a (or a^{-1}) will also be labeled P. Since the second vertex P is at the tail of edge b^{-1}, check for any other vertices at the tail of edges b or b^{-1}. Continue this process until all vertices P have been identified.

Choose an unlabeled vertex and call it Q. Repeat process used for vertex P. Continue with R, S, etc. until all vertices have been labeled.

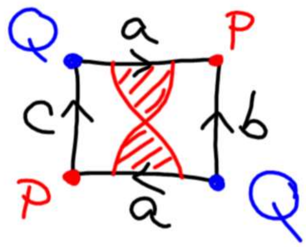


Since all "a" edges, "P" vertices, and "Q" vertices have been identified, we can now calculate the Euler characteristic, $\chi(S) = v - e + f$. We have 1 face, 3 edges, and 2 vertices, so $\chi(S) = 2 - 3 + 1 = 0$.

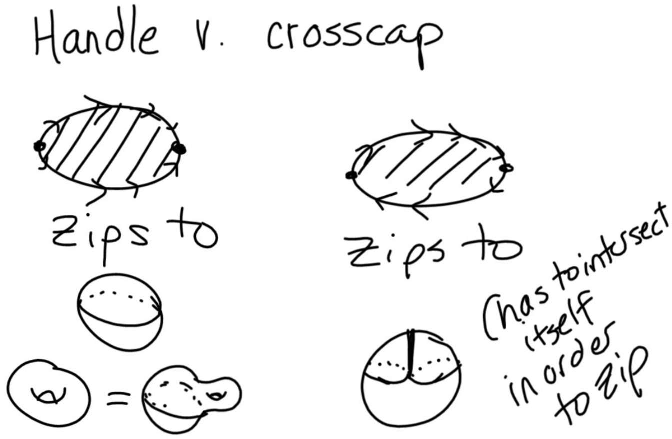
Next, we will take all unidentified edges (in this case, b & c) and determine the number of boundary components. Since b^{-1} and c both connect to vertices P & Q as shown, we have 1 boundary component.



Now, we can find the genus. Since the "a" edges have opposite orientation, we must put a twist (or Möbius band) in the surface to identify them. Any surface requiring one or more Möbius band is non-orientable.

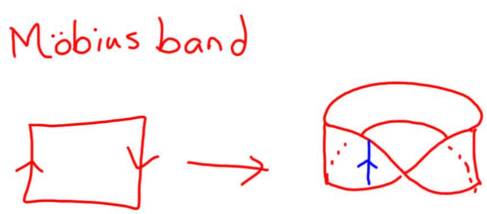


The genus of a surface refers to the number of handles (for orientable surfaces) or cross-caps (for non-orientable surfaces) present in the surface.

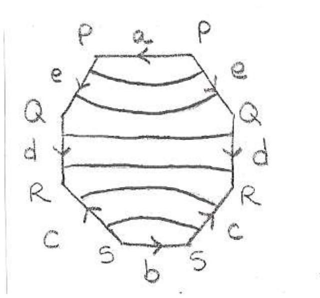


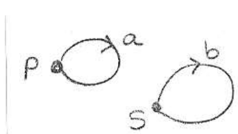
The formula relating Euler characteristic to genus for non-orientable surfaces is $\chi(S) = 2 - g - b$.

Here, $\chi(S) = 0$ and $b = 1$, so $0 = 2 - g - 1$ or $g = 1$. The word $ab^{-1}ac$ describes a non-orientable surface of genus 1 with 1 boundary component, the Möbius band.



Let's look at the word $aedc^{-1}bcd^{-1}e^{-1}$





1 face, 5 edges, 4 vertices → Euler characteristic $\chi(S) = v - e + f = 4 - 5 + 1 = 0$
No Möbius bands → orientable → $\chi(S) = 2 - 2g - b \rightarrow g = 0$
This is an orientable surface of genus 0 (no handles → sphere) with 2 boundary components, the annulus.