Trigonometry Notes – Sarah Brewer – Alabama School of Math and Science

Last Updated: 25 November 2011

6 Basic Trig Functions

Defined as ratios of sides of a right triangle in relation to one of the acute angles in the triangle.

Input of a trig function is an angle; output is a ratio of sides.

$$\mathbf{sin\,\theta} = \text{sine of } \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\mathbf{cos\,\theta} = \text{cosine of } \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\mathbf{tan\,\theta} = \text{tangent of } \theta = \frac{\text{opposite}}{\text{adjacent}}$$

These three trig functions can be recalled using the mnemonic SohCahToa.

The other three basic trig functions are reciprocals of the first three.

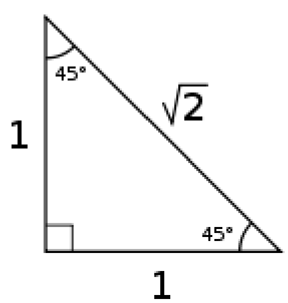
$$\frac{1}{\sin\theta} = \mathbf{csc\,\theta} = \text{cosecant of } \theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

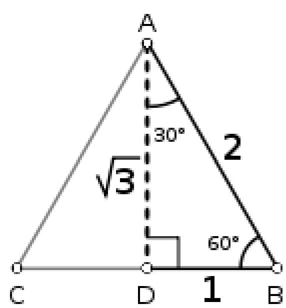
$$\frac{1}{\cos\theta} = \mathbf{sec\,\theta} = \text{secant of } \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\frac{1}{\tan\theta} = \mathbf{cot\,\theta} = \text{cotangent of } \theta = \frac{\text{adjacent}}{\text{opposite}}$$

Note that each pair of reciprocals only has one "co"

Special Triangles

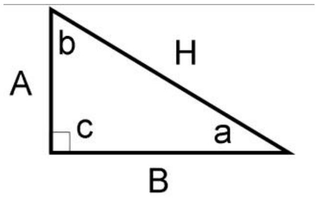




	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\csc\theta$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

Cofunctions

Note that what is the adjacent side for one of the acute angles in a right triangle is the opposite side for the other angle, and vice-versa.



Thus, any pair of trig functions where the only difference is "opposite" or "adjacent" are related by the cofunction identities.

Two functions f and g are called cofunctions if $f(90° - x) = g(x)$ and $g(90° - x) = f(x)$.

The cofunction identities are:

$\sin(90° - \theta) = \cos\theta$, $\cos(90° - \theta) = \sin\theta$

$\tan(90° - \theta) = \cot\theta$, $\cot(90° - \theta) = \tan\theta$

$\csc(90° - \theta) = \sec\theta$, $\sec(90° - \theta) = \csc\theta$

Note that pairs of cofunctions are just sin, cos, or tan, and the same word with co- in front of it.

Example Problem: Given that $\sin 23.7° \approx 0.4019$, $\cos 23.7° \approx 0.9157$, and $\tan 23.7° \approx 0.4390$, find the six trig functions of 66.3°.

$$\sin 66.3° = \cos(90° - 66.3°) = \cos 23.7° = \boxed{0.9157}$$

$$\cos 66.3° = \sin(90° - 66.3°) = \sin 23.7° = \boxed{0.4019}$$

$$\tan 66.3° = \cot(90° - 66.3°) = \cot 23.7° = \frac{1}{\tan 23.7°} = \frac{1}{0.4390} = \boxed{2.2779}$$

$$\csc 66.3° = \sec(90° - 66.3°) = \sec 23.7° = \frac{1}{\cos 23.7°} = \frac{1}{0.9157} = \boxed{1.0921}$$

$$\sec 66.3° = \csc(90° - 66.3°) = \csc 23.7° = \frac{1}{\sin 23.7°} = \frac{1}{0.4019} = \boxed{2.4882}$$

$$\cot 66.3° = \tan(90° - 66.3°) = \tan 23.7° = \boxed{0.4390}$$

Degree°Minute'Second'' Form

$1' = \frac{1}{60}°$ $\qquad$ $1'' = \frac{1}{60}' = \frac{1}{3600}°$

Example problems:

1. Convert $19°47'23''$ to decimal degrees to 2 decimal places.

$$19°47'23'' = 19° + \frac{47}{60}° + \frac{23}{3600}° = \boxed{19.79°}$$

2. Convert 20.14° to degrees, minutes, and seconds.

$$20.14° = 20° + .14 \cdot 1° = 20° + .14 \cdot 60' = 20° + 8.4' = 20° + 8' + .4 \cdot 1' = 20° + 8' + .4 \cdot 60'' =$$
$$= 20° + 8' + 24'' = \boxed{20°8'24''}$$

3. Convert $12°6'12''$ to decimal degrees to 3 decimal places.

$$12°6'12'' = 12° + \frac{6}{60}° + \frac{12}{3600}° = 12° + \frac{1}{10}° + \frac{1}{300}° = \boxed{12.103°}$$

4. Convert 14.26° to degrees, minutes, and seconds.

$$14.26° = 14° + .26 \cdot 60' = 14° + 15.6' = 14° + 15' + .6 \cdot 60'' = \boxed{14°15'36''}$$

Applications of Right Triangles

Angles of elevation and depression are always measured from the horizontal.

A bearing of $S24°E$ is read as "24 degrees east of south" – draw coordinate axes at the point from which the bearing is measured, start at south, and measure 24 degrees from the south axis toward the east

Example Problems

1. To an observer on the ground, the angle of elevation to a weather balloon is 45 degrees, and her distance to a spot on the ground directly below the balloon is 100 feet. How high is the balloon, in feet?

Solution: Since angles of elevation are measured from the horizontal, the angle from the ground to the hypotenuse of the triangle is $45°$, making this a $45° - 45° - 90°$ triangle. Hence, the height of the balloon is the same as her distance to a spot on the ground below the balloon, $\boxed{100\ feet}$.

2. A window washer on the side of building A measures his angle of elevation to the top of building B to be $60°$, and his angle of depression to the bottom of building B to be $45°$. If building A and building B are 100 feet apart, how tall is building B?

Solution: We split the height of building B into two parts (x and y) along the horizontal at the window washer, and set up two right triangles, of which we want to find the opposite sides. The equation for the top triangle is $\tan 60° = \frac{x}{100\ ft}$ and the equation for the bottom triangle is $\tan 45° = \frac{y}{100\ ft}$. Rearranging these equations, simplifying the trig expressions and setting $h = x + y$, we have that $\boxed{h = 100\sqrt{3} + 100\ ft}$.

Trigonometric Functions of Any Angle

An angle in standard position has its vertex at the origin and initial side on the positive x-axis, and is measured counterclockwise.

Two angles sharing a terminal side are called coterminal, and differ by integer multiples of 360°.

Example Problem: Find two positive and two negative angles that are coterminal with 89°.

$$89° + 360° = \boxed{449°} + 360° = \boxed{809°}$$

$$89° - 360° = \boxed{-271°} - 360° = \boxed{-631°}$$

Two acute angles are complementary if their sum is $90°$.

Two acute angles are supplementary if their sum is $180°$.

The acute angle formed by the terminal side of the angle in standard position and the x-axis is the reference angle. Any angles with the same reference angle will have the same trig function values, up to a sign change depending on the quadrant.

Trig functions of any angle can be found by finding the trig functions of the reference angle, taking into account whether the sides are positive or negative.

In the special case where the hypotenuse (radius) of the reference triangle is 1, we can see that the x-coordinate exactly corresponds to the sine of the angle, and the y-coordinate exactly corresponds to the cosine of the angle. Angles in the 1st and 4th quadrants will have a positive cosine value, angles in the 2nd and 3rd quadrants will have a negative cosine value, angles in the 1st and 2nd quadrants will have a positive sine value, and angles in the 3rd and 4th quadrants will have a negative sine value.

The other 4 trig functions can be written in terms of sine and cosine.

$$\sec x = \frac{1}{\cos x} \quad , \quad \csc x = \frac{1}{\sin x} \quad , \quad \tan x = \frac{\sin x}{\cos x} \quad , \quad \cot x = \frac{\cos x}{\sin x}$$

The mnemonic to remember which functions are positive in which quadrants is All Students Take Calculus: All functions are positive in the 1st quadrant, only sin and its reciprocal cosecant are positive in the 2nd quadrant, only tangent and its reciprocal cotangent are positive in the 3rd quadrant, and only cosine and its reciprocal secant are positive in the 4th quadrant.

Quadrantal angles are angles whose terminal side falls on one of the axes. Using the special case where the radius/hypotenuse is 1, and labeling points on each of the positive and negative x- and y-axes as $(1, 0), (0,1), (-1,0), and\ (0, -1)$ and thinking about cos as the x-coordinate and sin as the y-coordinate, and thinking of the other 4 trig functions in terms of sin and cos, we can see that all quadrantal angles (coterminal with either $0°, 90°, 180°, or\ 270°$) will have the following values:

	$0°$	$90°$	$180°$	$270°$
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	undefined	0	undefined
$\csc\theta$	undefined	1	undefined	-1
$\sec\theta$	1	undefined	-1	undefined
$\cot\theta$	undefined	0	undefined	0

Radians, Arc Length, & Angular Speed

The unit circle is a circle with center at the origin and radius 1: $x^2 + y^2 = 1$

The circumference of a circle is $2\pi r$, so for the unit circle, $c = 2\pi$.

We associate a new angle measure, called radians, equal to the length of the arc that subtends the specified angle.

$360° = 2\pi\ radians$

$180° = \pi\ radians$

$\frac{180°}{\pi} = 1$, $\frac{\pi}{180°} = 1$

To convert from radians to degrees, multiply by $\frac{180°}{\pi}$

To convert from degrees to radians, multiply by $\frac{\pi}{180°}$

Note that no units is the same as radians, even if there is no π present.

Variables

$s = distance\ traveled$ or $arc\ length\ (inches, kilometers, etc)$

$t = time\ (seconds, minutes, hours, etc)$

$\theta = amount\ of\ rotation$ or $included\ angle\ (degrees, radians, rotations, revolutions, etc)$

$r = radius\ or\ distance\ from\ the\ center\ of\ rotation\ (centimeters, inches, etc)$

$$v = linear\ speed = \frac{distance}{time}$$

$$\omega = angular\ speed = \frac{amount\ of\ rotation}{time}$$

Formulas

$$s = r\theta\ , \quad v = \frac{s}{t}\ , \quad \omega = \frac{\theta}{t}\ , \quad v = r\omega$$

Note: We derive $v = r\omega$ by combing $v = \frac{s}{t}$ with $s = r\theta$ to get $v = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t} = r\omega$

Dimensional analysis conversion factors

$$\frac{5280\ ft}{1\ mi}, \frac{12\ in}{1\ ft}, \frac{2\pi}{1\ rev}, \frac{\pi}{180°}, \frac{60\ min}{1\ hr}, \frac{60\ sec}{1\ min}, and\ their\ reciprocals$$

Steps to Solve

1. Identify what is given and what you are trying to find; identify all variables and associated units.
2. Determine which equation relates the known and unknown variables.
3. Rearrange the equation to solve for the unknown variable.
4. Plug in known quantities with units into equation.
5. Multiply by a series of dimensional analysis conversion factors until you arrive at the appropriate units for your answer.

Example Problems

1. A wheel with a 15-inch diameter rotates at a rate of 6 radians per second. What is the linear speed of a point on its rim in feet per minute?

Solution: $r = \frac{15}{2} in\ ,\ \omega = 6\frac{rad}{s}\ ,\ v = ?\,ft/min$ Equation: $v = r\omega$

$$v = r\omega = \frac{15\ in}{2} \cdot \frac{6\ rad}{1\ s} \cdot \frac{1\ ft}{12\ in} \cdot \frac{60\ s}{1\ min} = \boxed{225\ ft/min}$$

2. An earth satellite in circular orbit 1200 km high makes one complete revolution every 90 minutes. What is its linear speed in km/min, given that the earth's radius is 6400 km?

Solution: $r = 6400 + 1200 = 7600\ km\ ,\ \omega = \frac{1\ rev}{90\ min}\ ,\ v = ?\,km/min$ Equation: $v = r\omega$

$$v = r\omega = \frac{7600km}{1} \cdot \frac{1\ rev}{90\ min} \cdot \frac{2\pi}{1\ rev} = \frac{760 \cdot 2\pi\ km}{9\ min} = \boxed{\frac{1520\pi}{9} km/min}$$

3. Through how many radians does the minute hand of a clock rotate from 12:45pm to 1:25pm?

Solution: $\theta = ?,\ t = 40\ min,\ \omega = \frac{2\pi}{1\ hr}$ Equation: $\omega = \frac{\theta}{t}$

$$\theta = \omega t = \frac{2\pi}{1\ hr} \cdot \frac{40\ min}{1} \cdot \frac{1\ hr}{60\ min} = \boxed{\frac{4\pi}{3}}$$

4. A car travels at 60 miles per hour. Its wheels have a 24-inch diameter. What is the angular speed of a point on the rim of a wheel in revolutions per minute?

Solution: $v = \frac{60\ mi}{1\ hr}, = 12\ in, \omega = ?\frac{rev}{min}$ Equation relating these variables: $v = r\omega$

$$\omega = \frac{v}{r} = v \cdot \frac{1}{r} = \frac{60\ mi}{1\ hr} \cdot \frac{1}{12\ in} \cdot \frac{1\ hr}{60\ min} \cdot \frac{5280\ ft}{1\ mi} \cdot \frac{12\ in}{1\ ft} \cdot \frac{1\ rev}{2\pi} = \boxed{\frac{2640}{\pi} \frac{rev}{min}}$$

5. A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

Solution: $s = 3mi,\ \theta = 2640\ rev,\ r = ?\,in$ Equation: $s = r\theta$

$$r = \frac{s}{\theta} = \frac{3\ mi}{2640\ rev} \cdot \frac{5280\ ft}{1\ mi} \cdot \frac{12\ in}{1\ ft} \cdot \frac{1\ rev}{2\pi} = \boxed{\frac{36}{\pi} in}$$

6. A satellite 290 miles above Mars' surface makes one revolution every 2 hours. What is its linear speed in miles per hour, given that the radius of Mars is 2110 miles?

Solution: $r = 290\ mi + 2110\ mi = 2400\ mi,\ \omega = \frac{1\ rev}{2\ h},\ v = ?\frac{mi}{h}$, Equation: $v = r\omega$

$$v = r\omega = \frac{2400\ mi}{1} \cdot \frac{1\ rev}{2\ h} \cdot \frac{2\pi}{1\ rev} = \boxed{2400\pi\ \frac{mi}{h}}$$

7. A pulley has a 48-inch diameter, and moves a belt at a rate of 8 miles per hour. What is the angular speed of a point on the edge of the pulley in revolutions per minute?

Solution: $r = 24\ in,\ v = \frac{8mi}{h}, \omega = ?\frac{rev}{min},$ Equation: $v = r\omega$

$$\omega = \frac{v}{r} = v \cdot \frac{1}{r} = \frac{8mi}{h} \cdot \frac{1}{24\ in} \cdot \frac{5280\ ft}{1\ mi} \cdot \frac{12\ in}{1\ ft} \cdot \frac{1\ h}{60\ min} \cdot \frac{1\ rev}{2\pi} = \boxed{\frac{176\ rev}{\pi\ \ min}}$$

8. A circle has a radius of 3 feet. What is the measure, in degrees, of an angle that subtends an arc of 4 inches?
Solution: $r = 3\ ft,\ s = 4\ in,\ \theta = ?°$, Equation: $s = r\theta$

$$\theta = \frac{s}{r} = \frac{4\ in}{3\ ft} \cdot \frac{1\ ft}{12\ in} \cdot \frac{180°}{\pi} = \boxed{\frac{20°}{\pi}}$$

9. The angle of depression to the bottom of a slide is 60 degrees. If a child slides down at a rate of 5 feet per second and it takes 2 seconds for the child to reach the bottom, what is the vertical height of the slide, in feet?
Solution: Note that here that our 60° is an angle of depression and NOT an amount of rotation. This is a fixed angle, measured from the horizontal. If you draw a picture of the slide, you will see that, due to the rule of two parallels cut by a transversal, the acute angle between the ground and the slide is 60° as well. We want to find the height of the slide. Using that $v = 5\frac{ft}{s}$ and $t = 2\ s$, we can see that the length of the slide, $s = 10ft$ (using the formula $v = \frac{s}{t}$). The rest is just an application of right triangles problem. $\sin 60° = \frac{h}{10\ ft}$, so $h = 10 \sin 60° = \boxed{5\sqrt{3}ft}$

Graphing

The domain of a function is the set of all x-values that make sense when plugged into the function.
The range of a function is the output of the domain, or the set of all possible y-values.
The period of a function is the smallest interval over which it repeats itself.
A function is even if $f(-x) = f(x)$. (Plugging in –x for x yields the original function.) Even functions are symmetric with respect to the y-axis. Even trig functions include: $\cos x$, $\sec x$
A function is odd if $f(-x) = -f(x)$. (Plugging in –x for x yields the negative of the original function). Odd functions are symmetric with respect to the origin. Odd trig functions include: $\sin x$, $\csc x$, $\tan x$, $\cot x$

Function	Domain	Range	Period
$y = \sin x$	$(-\infty, \infty)$	$[-1,1]$	2π
$y = \cos x$	$(-\infty, \infty)$	$[-1,1]$	2π
$y = \csc x$	$\{x \mid x\ is\ not\ an\ integer\ multiple\ of\ \pi\}$	$(-\infty, -1] \cup [1, \infty)$	2π
$y = \sec x$	$\left\{x \mid x\ is\ not\ an\ odd\ multiple\ of\ \frac{\pi}{2}\right\}$	$(-\infty, -1] \cup [1, \infty)$	2π
$y = \tan x$	$\left\{x \mid x\ is\ not\ an\ odd\ multiple\ of\ \frac{\pi}{2}\right\}$	$(-\infty, \infty)$	π
$y = \cot x$	π	$(-\infty, \infty)$	π

See "Trigonometry Guide to Graphing" for details about how to graph transformations of the 6 basic trig functions, as well as how to graph sums and differences of trig functions.

Identities

An identity is an equation that is true for all elements in its domain.
Identities to know:

Reciprocal Identities

$\csc x = \frac{1}{\sin x}$, $\sin x = \frac{1}{\csc x}$, $\sec x = \frac{1}{\cos x}$, $\cos x = \frac{1}{\sec x}$, $\cot x = \frac{1}{\tan x}$, $\tan x = \frac{1}{\cot x}$

Ratio Identities

$\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$

Odd-Even Identities

$\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, $\tan(-x) = -\tan x$

$\sec(-x) = \sec x$, $\csc(-x) = -\csc x$, $\cot(-x) = -\cot x$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$, $1 + \cot^2 x = \csc^2 x$, $\tan^2 x + 1 = \sec^2 x$

Sum and Difference Identities

$\sin(a+b) = \sin a \cos b + \cos a \sin b$, $\sin(a-b) = \sin a \cos b - \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$, $\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$, $\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$, $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x$, $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

$\csc\left(\frac{\pi}{2} - x\right) = \sec x$, $\sec\left(\frac{\pi}{2} - x\right) = \csc x$

Double-Angle Identities

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

Half-Angle Identities

$\sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$, $\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$

$\tan\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$

Tips for proving identities:

- Simplify more complex side until it is equal to the other
- add fractions, square binomials, factor
- use known identities
- rewrite in terms of sines and cosines
- rewrite in terms of a single trig function
- rewrite numerator and denominator by the same factor (1), often the conjugate of one of them
- keep goal in mind
- if you think it's not an identity, find a single value that doesn't work

Example proofs:

1. $\cos 5x \cos 3x + \sin 5x \sin 3x = \cos^2 x - \sin^2 x$

Proof:

$LHS = \cos(5x - 3x) = \cos 2x = \cos^2 x - \sin^2 x = RHS$

2. $\sin(a-b) - \sin(a+b) = -2\cos a \cos b$

Proof:

$LHS = (\sin a \cos b - \cos a \sin b) - (\sin a \cos b + \cos a \sin b) = -2 \cos a \sin b = RHS$

3. $\cos 3x = 4\cos^3 x - 3\cos x$

Proof:

$LHS = \cos(2x + x) = \cos 2x \cos x - \sin x \sin 2x = (2\cos^2 x - 1) \cdot \cos x - \sin x \cdot (2 \sin x \cos x) =$

$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x = 2\cos^3 x - \cos x - 2(1 - \cos^2 x) \cdot \cos x =$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x = 4\cos^3 x - 3\cos x = RHS$

4. $\sin 4x = 4 \sin x \cos^3 x - 4 \cos x \sin^3 x$

Proof:
$LHS = \sin 2(2x) = 2\sin 2x \cos 2x = 2(2\sin x \cos x)(\cos^2 x - \sin^2 x) =$
$= 4\sin x \cos^3 x - 4\cos x \sin^3 x = RHS$

5. $\cos^2 \frac{x}{2} \sec x = \frac{1}{2}(\sec x + 1)$

Proof:

$$LHS = \left(\cos\frac{x}{2}\right)^2 \sec x = \left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 \sec x = \frac{1+\cos x}{2} \cdot \sec x = \frac{\sec x + \cos x \sec x}{2} =$$
$$= \frac{\sec x + \cos x \cdot \frac{1}{\cos x}}{2} = \frac{1}{2}(\sec x + 1) = RHS$$

6. $\frac{2\cos 2x}{\sin 2x} = cot\ x - \tan x$

Proof:

$$LHS = \frac{2(\cos^2 x - \sin^2 x)}{2\sin x \cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{\cos^2 x}{\sin x \cos x} - \frac{\sin^2 x}{\sin x \cos x} = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} =$$
$= cot\ x - \tan x = RHS$

Inverse Trigonometric Functions

Background on Inverse Functions:

A function f is one-to-one if $f(a) = f(b)$ implies that $a = b$. Graphically, a one-to-one function passes both the vertical and horizontal line tests. One y for each x and one x for each y.

If f is a one-to-one function with domain X and range Y, and g is a function with domain Y and range X, then g is the inverse function of f if and only if:

$$(f \circ g)(x) = x \ for\ all\ x\ in\ the\ domain\ of\ g$$

$$(g \circ f)(x) = x \ for\ all\ x\ in\ the\ domain\ of\ f$$

That is, inverse functions "undo" each other.

Inverse Trig functions:

Trig functions in general are not one-to-one, so in order to define their inverses, we must first restrict the domains of the original functions. The following table lists all of the trig functions and their inverses, their corresponding restricted domains, and the range of each function.

function	domain	range	function	domain	range
$\boldsymbol{y = \sin x}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1,1]$	$\boldsymbol{y = \sin^{-1} x}$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\boldsymbol{y = \cos x}$	$[0, \pi]$	$[-1,1]$	$\boldsymbol{y = \cos^{-1} x}$	$[-1,1]$	$[0, \pi]$
$\boldsymbol{y = \tan x}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$	$\boldsymbol{y = \tan^{-1} x}$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\boldsymbol{y = \csc x}$	$\left[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}\right]$	$(-\infty, -1] \cup [1, \infty)$	$\boldsymbol{y = \csc^{-1} x}$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}\right]$
$\boldsymbol{y = \sec x}$	$[0, \frac{\pi}{2}) \cup \left(\frac{\pi}{2}, \pi\right]$	$(-\infty, -1] \cup [1, \infty)$	$\boldsymbol{y = \sec^{-1} x}$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup \left(\frac{\pi}{2}, \pi\right]$
$\boldsymbol{y = \cot x}$	$(0, \pi)$	$(-\infty, \infty)$	$\boldsymbol{y = \cot^{-1} x}$	$(-\infty, \infty)$	$(0, \pi)$

Note that the -1 exponent indicates inverse NOT reciprocal. $\sin^{-1} x$ is <u>not</u> the same as $\frac{1}{\sin x}$.
Also note that the domains and ranges are swapped. The input for a basic trig function is an angle and its output is a ratio of sides. For an inverse trig function, its input is a ratio of sides and its output is an angle.

<u>Evaluating Inverse Trig Functions:</u>

To evaluate the expression $\cos^{-1}\left(\frac{1}{2}\right)$, we ask the question: What angle θ between 0 and π is such that $\cos\theta = \frac{1}{2}$?
Since cosine is defined as the adjacent side over the opposite side, we know that the reference angle must be $\frac{\pi}{3}$. Also, because $\frac{1}{2}$ is positive (and we know that cosine is positive in the 1st and 4th quadrants), we know that the angle must either be in the 1st or 4th quadrants. The interval $[0, \pi]$ spans the 1st and 2nd quadrants, so we know that the angle must be in the 1st quadrant. Hence, the solution is $\frac{\pi}{3}$.

Next, let's look at the expression $\csc^{-1}(-2)$. We want the angle θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $\csc\theta = -2$. Since cosecant is defined as hypotenuse over opposite, and we can think of -2 as $-\frac{2}{1}$, we know that the reference angle is $\frac{\pi}{6}$.
We also know that cosecant is negative in the 3rd and 4th quadrants. The interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ spans the 4th and 1st quadrants, so we know that the angle must be in the 4th quadrant. So, we're looking for the $\frac{\pi}{6}$ reference angle in the 4th quadrant that is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. One might be tempted to say $\frac{11\pi}{6}$, but this number is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
We want the angle coterminal with $\frac{11\pi}{6}$ that is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Hence, the solution is $-\frac{\pi}{6}$.

<u>Compositions of Trig Functions:</u>

For all x in the domain of the inside function, $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. This holds for trig functions composed with their inverses as well.

First, let's look at the expression $\cot^{-1}\left(\cot\frac{2\pi}{3}\right)$. Since $\frac{2\pi}{3}$ is in the interval $[0, \pi]$, the solution is $\frac{2\pi}{3}$.
Now, let's look at $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$. $\frac{2\pi}{3}$ is NOT in the restricted domain of the sine function, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, we need to find an angle that is in the restricted domain that has the same sine value as $\frac{2\pi}{3}$. $\frac{2\pi}{3}$ is in the 2nd quadrant, where the sine function is positive, so we want to look at the 1st quadrant for our solution, since sine is negative in the 4th quadrant. The angle with the same sine value as $\frac{2\pi}{3}$ will necessarily have the same reference angle, $\frac{\pi}{3}$. So, the solution is the angle in the first quadrant with a $\frac{\pi}{3}$ reference angle, $\frac{\pi}{3}$ itself.

One thing to keep in mind about these last two problems is that the outermost functions in each compositions were inverse functions, which told us that our solution or output needed to be an angle. If the outermost function of a composition is a regular trig function, our solution should be a ratio of sides.

When the inner function is an inverse and the outer function is a basic trig function, it is easier to evaluate, because the answer to, for example, $\sin(\sin^{-1} x)$ will always be x unless $\sin^{-1} x$ is undefined, in which case the entire expression is undefined.

In composing different trig functions, I tend to look at the outermost function first to determine if my final answer is going to be an angle or a ratio of sides, and usually draw a picture in the appropriate quadrant, taking into account restricted domain and whether the function value is positive or negative.

Solving Trigonometric Equations

Solving a trig equation is much like solving an algebraic equation, except there may be some identities and inverse trig functions involved in order to solve for x. The key thing to remember here is that unlike evaluating the expression $\sin^{-1}\left(\frac{1}{2}\right)$ (for example) which only has one solution as determined by the restricted domain for $\sin x$, when solving the equation $\sin x = \frac{1}{2}$, you are no longer dealing with a restricted domain unless specifically stated. In fact, there are infinitely many solutions to this equation. If we look at solutions between 0 and 2π, we can see that the two angles in quadrants where sine is positive whose reference triangles have opposite side length 1 and hypotenuse length 2 are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. All other solutions will be coterminal with these two, and so we can describe our solution set as $\left\{\frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k \mid k\ is\ an\ integer\right\}$. If we were just looking for solutions in the interval $[0, 2\pi)$ (as is often the case; note here that 0 is included but 2π is not -- since the two angles are coterminal we only want to include one of them), our solution set would simply be $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

Often we can't simplify our equation to something as simple as $\sin x = \frac{1}{2}$. Sometimes we may end up with something like $\sin 3x = \frac{1}{2}$. In this case, it is 3x that is equal to the values in our solution set rather than x, so all of those values in the solution set have to be divided by 3, making our solution set $\left\{\frac{\pi}{18} + \frac{2\pi k}{3}, \frac{5\pi}{18} + \frac{2\pi k}{3} \mid k\ is\ an\ integer\right\}$. If we are looking for solutions strictly in the interval $[0, 2\pi)$, our job is slightly more difficult. Here, we want $0 \le x < 2\pi$, which means we need $0 \le 3x < 6\pi$. If we only took our solution set to be the two angles $\frac{\pi}{18}, \frac{5\pi}{18}$, we would not be accounting for all of the other solutions up to 6π. So, we approach the problem like this: 6π is 3 times around the unit circle. Note that 3 is the number we have in front of our x. This will always be the case - the number in front of x tells us how many times to go around the unit circle for solutions. We start with $3x = \frac{\pi}{6}, \frac{5\pi}{6}$ and before dividing by 3, we are going to list all other angles coterminal to those 2 up to 6π by adding multiples of 2π, in this case of the form $\frac{12\pi}{6}$. We end up with $3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$. Now, divide both sides by 3 to solve for x. $x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$.

The above methods are the only thing new here. Knowledge of the unit circle, trig identities, and basic algebraic techniques of solving equations make up the bulk of these problems. The algebraic techniques most often used will be the Zero Product Property ($if\ AB = 0, then\ A = 0\ or\ B = 0$) and the Square Root Theorem $\left(if\ f^2(x) = A, then\ f(x) = \pm\sqrt{A}\right)$.

Simple examples:

$1.\ 2\sin x = \sqrt{3}, \qquad x \in [0, 2\pi)$

Solution: We start by isolating $\sin x$ by dividing both sides by 2 to get $\sin x = \frac{\sqrt{3}}{2}$. Since we only want solutions between 0 and 2π, we know that we will have 2 solutions, one in each of the quadrants where sine is positive. Since the sine value is $\frac{\sqrt{3}}{2}$, we know that we have a $\frac{\pi}{3}$ reference angle. The solution set then is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$.

$2.\ 4\cos^3 x = 3\cos x, \qquad x \in [0, 2\pi)$

Solution: Here, an important thing to note is that if we were to divide both sides by $\cos x$, we would lose some of our solutions (just as dividing the equation $x^2 = 2x$ by x causes us to lose the solution $x = 0$), so it is important that we bring everything over to one side of the equation and instead FACTOR out $\cos x$ to get $\cos x\,(4\cos^2 x - 3) = 0$. Now we can use the Zero Product Property to set each of our factors equal to 0, which breaks our equation up into two

smaller equations: $\cos x = 0$ or
$4\cos^2 x - 3 = 0$. The solution to the first equation is $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$.

Example equations where there is a coefficient in front of x:

Example 1. Solve for x. (no restrictions on $x \to$ infinitely many solutions) $\mathbf{\tan 3x = -1}$
Solution: $3x$ is the angle whose tangent value is -1. There are infinitely many such angles, so we start with those the first time around the unit circle. A tangent value of 1 or -1 implies adjacent and opposite side lengths are the same, so we are looking at a $45°$ or $\frac{\pi}{4}$ reference angle. Tangent is positive in quadrants I & III and negative in quadrants II & IV, so we want the angles in quadrants II and IV with $\frac{\pi}{4}$ reference angles. This yields: $3x = \frac{3\pi}{4}$ and $3x = \frac{7\pi}{4}$
Remember, though, that we are looking for ALL such angles, and they will all be coterminal with these two, so we add $2\pi k$, where k is an integer, to each solution to get $3x = \frac{3\pi}{4} + 2\pi k$ and $3x = \frac{7\pi}{4} + 2\pi k$
We want to solve for x rather than $3x$, so we now have to divide both sides of each equation by 3 to get

$$\boxed{x = \frac{\pi}{4} + \frac{2\pi k}{3} \text{ and } x = \frac{7\pi}{12} + \frac{2\pi k}{3}}$$

This is a perfectly acceptable solution. Notice however, that the angles $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ differ exactly by π, so rather than writing $3x = \frac{3\pi}{4} + 2\pi k$ and $3x = \frac{7\pi}{4} + 2\pi k$, we could simply write the single expression
$3x = \frac{3\pi}{4} + \pi k$, which would cover all of the same angles as the two former expressions. This yields a much nicer, simpler solution: $\boxed{x = \frac{\pi}{4} + \frac{\pi k}{3}}$.

Example 2. Solve for x. (no restrictions on $x \to$ infinitely many solutions) $\mathbf{\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}}$
Solution: $2x - \frac{\pi}{6}$ is the angle whose cosine value is $\frac{\sqrt{3}}{2}$, and again, there are infinitely many such angles. The first time around the unit circle, they are the $\frac{\pi}{6}$ reference angles in quadrants I & IV, where cosine is positive. This gives us
$2x - \frac{\pi}{6} = \frac{\pi}{6} + 2\pi k$ and $2x - \frac{\pi}{6} = \frac{11\pi}{6} + 2\pi k$. Solving for x by adding $\frac{\pi}{6}$ to both sides and then dividing by 2, we get
$2x = \frac{\pi}{6} + \frac{\pi}{6} + 2\pi k$ and $2x = \frac{11\pi}{6} + \frac{\pi}{6} + 2\pi k$
$2x = \frac{2\pi}{6} + 2\pi k$ and $2x = \frac{12\pi}{6} + 2\pi k$
$2x = \frac{\pi}{3} + 2\pi k$ and $2x = 2\pi + 2\pi k$

$$\boxed{x = \frac{\pi}{6} + \pi k \text{ and } x = \pi + \pi k}$$

The last equation can be written more simply as $x = \pi k$, since $\pi = \pi k$ when $k = 1$.

$$\boxed{x = \frac{\pi}{6} + \pi k \text{ and } x = \pi k}$$

Example 3. Solve for $x \in [0, 2\pi)$. (note the restriction on x) $\mathbf{\sin 4x = \frac{1}{2}}$
Solution: We start by finding the angles the first time around the unit circle with a sine value of $\frac{1}{2}$. An adjacent side of 1 and hypotenuse of 2 tells us these have $\frac{\pi}{6}$ reference angles, and sine is positive in quadrants I & II, so we set
$4x = \frac{\pi}{6}, \frac{5\pi}{6}$
Since we're looking for $0 \le x < 2\pi$, we need $0 \le 4x < 8\pi$, i.e. we're looking for solutions 4 times around the unit circle so that when we divide by 4 to solve for x, we end up with all of the possible solutions between 0 and 2π. We accomplish this by adding 2π to each solution for the 2nd time around the unit circle, again for the third, and again for the fourth. Adding 2π to $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ requires getting a common denominator, so we rewrite $2\pi \cdot \frac{6}{6} = \frac{12\pi}{6}$, so we're adding

$\frac{12\pi}{6}$ to each solution three times (as opposed to adding 2π infinitely many, or k, times as we did in the previous examples). This yields $4x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}$

Finally, dividing each solution by 4 is the same as multiplying by $\frac{1}{4}$ (or multiplying the numerators by 1 and the denominators by 4), so our final solution set is

$$\boxed{x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{37\pi}{24}, \frac{41\pi}{24}}$$

Example 4. Solve for $x \in [0,2\pi)$. (note the restriction on x) $\mathbf{\tan 2x = 0}$

Solution: Tangent can always be thought of as sine over cosine, and a fraction is equal to zero wherever its denominator is equal to zero, so $\tan x = 0$ the same places $\sin x = 0$. The first time around the unit circle these are at 0 and π. Then we add 2π to each of these solution once for a total of two trips around the unit circle (the coefficient in front of x always tells you how many times to go around), and then divide by 2.

$2x = 0, \pi, 2\pi, 3\pi$

$$\boxed{x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}}$$

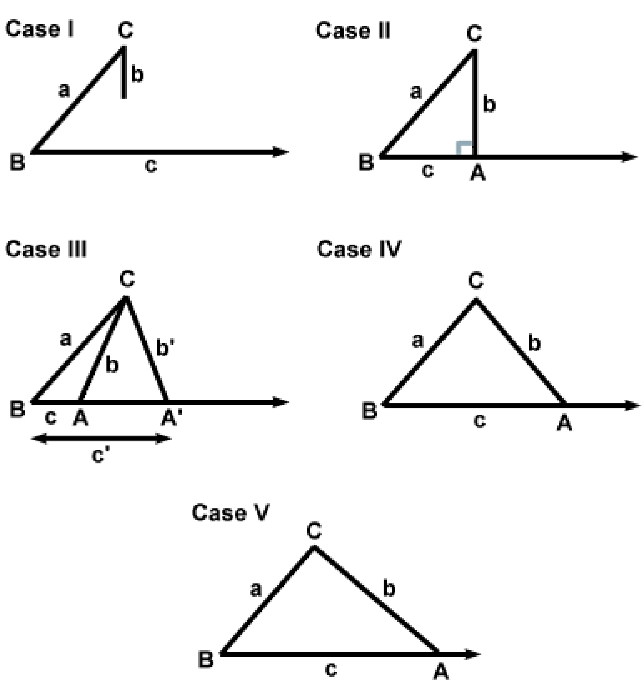
Solving Triangles

To solve a triangle means to find all of the angles and side lengths of the triangle. Using the 6 basic trigonometric functions, we can solve any right triangle. If the triangle is not right, or oblique, we need to use either the Law of Sines or the Law of Cosines.

The Law of Sines states that the ratios of an angle to an opposite side are all equal.

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ or $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

We use the Law of Sines to solve ASA, AAS, and SSA triangles. The latter we may want to refer to as an ASS triangle instead, because it is problematic. The two sides and angle given in a ASS triangle are one of four types:



Cases II. IV, and V each only have one possible solution. Case I has no solution. Case III has 2 possible solutions, since side b could be positioned either one of two different ways, as shown in the diagram.

The Law of Cosines is used to solve SSS and SAS triangles. There are three different versions of the Law of Cosines, but notice that in all three, the side on the left hand side of the equation corresponds to the angle on the right hand side.
$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

Vectors

A vector is a directed line segment, completely determined by its direction angle and magnitude (length).

Because a vector is uniquely determined by direction angle and magnitude, its location does not matter. We call any two vectors with the same direction angle and magnitude equivalent. Typically, we like to work with vectors whose initial points (tails) are at the origin so that we can describe them by their terminal points (heads). The terminal point of an equivalent vector whose initial point is at the origin can be found by subtracting the initial point from the terminal point.

e.g. Given a vector whose initial point is $(-5, 3)$ and terminal point is $(2, -6)$, the terminal point of an equivalent vector whose initial point is at the origin is found by $(2-(-5), -6-3) = (7, -9)$

Note that the parenthetical notation refers only to the terminal point of the vector, not the vector itself. To describe the vector a vector whose initial point is $(0,0)$ and terminal point is (a, b), we use $\vec{v} = \langle a, b \rangle$

The magnitude, or length of a vector is found by $\boxed{|\vec{v}| = \sqrt{a^2 + b^2}}$

Magnitude is an example of a scalar quantity, that is, a number without associated direction. A vector is a number with an associated direction.

To find the direction angle of a vector (measured counter-clockwise from 0), first we must find the corresponding reference angle $\alpha = \left|\tan^{-1}\frac{b}{a}\right|$. We use absolute value since a reference angle is always positive. The direction angle θ is found according to which quadrant the angle is in. In the first quadrant, $\theta = \alpha$; in Q2, $\theta = 180° - \alpha$; in Q3, $\theta = 180° + \alpha$, and in Q4, $\theta = 360° - \alpha$.

e.g.

Unit Vectors
A unit vector is a vector whose magnitude is 1. Two special unit vectors are $\vec{i} = \langle 1,0 \rangle$ and $\vec{j} = \langle 0,1 \rangle$. Any vector $\vec{v}$ can be written in terms of $\vec{i}$ and $\vec{j}$, but in order to do that, we need to use

Scalar Multiplication and Vector Addition/Subtraction
Given vectors $\vec{v} = \langle a, b \rangle$ and $\vec{w} = \langle c, d \rangle$,
$k\vec{v} = \langle ka, kb \rangle$
$\vec{v} \pm \vec{w} = \langle a \pm c, b \pm d \rangle$

e.g.

How to write a vector in terms of $\vec{i}$ and $\vec{j}$
Applying the above concepts of scalar multiplication and vector addition/subtraction, we can see that

$\langle -3,5\rangle = \langle -3,0\rangle + \langle 0,5\rangle = -3\langle 1,0\rangle + 5\langle 0,1\rangle = -3\vec{\imath} + 5\vec{\jmath}$

Any vector $\vec{v} = \langle a, b\rangle$ can be written as $\vec{v} = a\vec{\imath} + b\vec{\jmath}$

How to find a unit vector $\vec{u}$ in the direction of a given vector $\vec{v}$

Given a vector $\vec{v} = \langle a, b\rangle$ whose magnitude is $|\vec{v}| = \sqrt{a^2 + b^2}$, we can find a unit vector $\vec{u}$ in the direction of $\vec{v}$ by simply diviting each component by the magnitude.

$$\boxed{\vec{u} = \langle \frac{a}{|\vec{v}|}, \frac{b}{|\vec{v}|}\rangle}$$

e.g.

Dot Product (Vector Multiplication)

Given vectors $\vec{v} = \langle a, b\rangle$ and $\vec{w} = \langle c, d\rangle$,

$$\boxed{\vec{v} \cdot \vec{w} = ac + bd}$$

Note that the dot product is a scalar quantity.

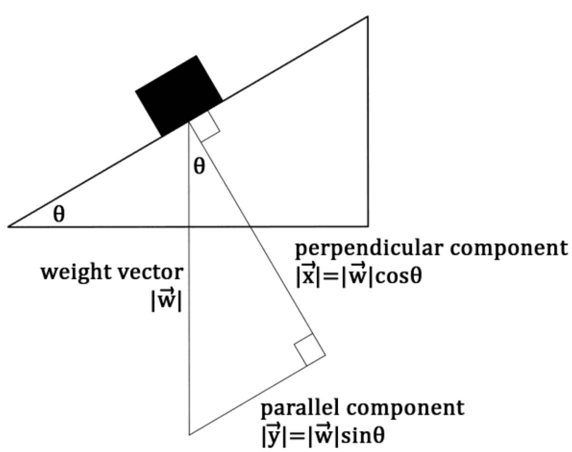
e.g.

The smallest nonnegative angle between two vectors is given by

$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$, or equivalently, $\boxed{\theta = \cos^{-1}\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}}$.

Parallel and Perpendicular (Normal) Components of a Force

The object on a ramp problem: Suppose we have an object of weight $\vec{w}$ at rest on a ramp with an incline of θ degrees. The weight vector $\vec{w}$ will always point straight down. We can split that vector into two components, one parallel to the ramp, and one perpendicular (or normal) to the ramp. The angle between the weight vector $\vec{w}$ and the normal component $\vec{x}$ will be the same as the angle of inclination of the ramp, because we are forming similar triangles. Note that the angle between the normal component $\vec{x}$ and the parallel component $\vec{y}$ will always be 90°. Since $\sin\theta = \frac{opp}{hyp}$ and $\cos\theta = \frac{adj}{hyp}$, the magnitude of the normal component is $|\vec{x}| = |\vec{w}|\cos\theta$ and the magnitude of the parallel component is $|\vec{y}| = |\vec{w}|\sin\theta$.



<u>Trigonometric Form of Complex Numbers</u>

Recall that a complex number z is a number of the form $z = a + bi$, where $i = \sqrt{-1}$, a is the "real component" and b is the "imaginary component." Rather than giving a complex number in $a + bi$ or "Standard" form, we can give it in trigonometric form. If we think of the a and b like the components of a vector, we can associate with each complex number an angle and a magnitude. $|z| = \sqrt{a^2 + b^2}$, just as if z were a vector, and the direction angle θ is found the same way - counterclockwise from the positive x-axis. The magnitude is referred to as the <u>modulus</u> r of the complex number, and the direction angle is referred to as the <u>argument</u> of the complex number. We can see that the real component $a = r\cos\theta$ and the imaginary component $b = r\sin\theta$. Hence, we can write a complex number $z = r\cos\theta + i \cdot r\sin\theta$. This is often shortened to $z = r(\cos\theta + i\sin\theta)$ or $z = r\operatorname{cis}\theta$.

Trigonometric form of complex numbers is especially useful for multiplying and dividing complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] = r_1 r_2 \operatorname{cis}\theta$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] = \frac{r_1}{r_2} \operatorname{cis}\theta$$

We can see that when multiplying complex numbers in trigonometric form, we multiply the moduli and add the arguments. To divide complex numbers in trigonometric form, we divide the moduli and subtract the arguments.

To raise a number to a power n is the same as multiplying it by itself n times. Hence,

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] = r^n \operatorname{cis}(n\theta)$$